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A two-warehouse inventory model for non-instantaneous deteriorating items over stochastic planning horizon

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ABSTRACT

This paper deals with a two-warehouse inventory control problem for non-instantaneous deteriorating items in stochastic framework, wherein shortages are permissible and are mixture of partially backlog and lost sales. The paper tackles a situation in which the retailer procures more quantity than his/her own warehouse (OW) storage capacity. In such a situation, a rented warehouse (RW) is needed to keep the over purchasing. The stochastic framework captures the situation of random planning horizon of trading. The multifariousness of the random planning horizon is discussed by considering two special cases, namely, uniform and truncated normal distributions. Depending upon the consumption times t_o and t_r of OW and RW inventories, respectively, and preserve time t_p after that product starts to deteriorate, we formulate mathematical models for three cases: (i) $t_p \le t_r \le t_o$, (ii) $t_r \le t_p \le t_o$, and (iii) $t_r \le t_o \le t_p$. The discussion is further elongated by presenting some numerical illustrations with comprehensive sensitivity analysis for changes in the value of parameters.

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Inventory; random time horizon; two-warehouse; non-instantaneous deterioration; partial backlogging

1. Introduction

In many marketplaces, like super market and municipality market, the space constraint is quite relevant because retail outlets have limited shelf space. A retailer warehouse is usually situated near to a shop which is generally located in urban area. Despite it, market scenarios such as relatively high ordering cost that precludes the small ordering size, price discount, stock against supply disruptions, and location of market enforce a retailer to procure the product beyond his/her own warehouse (OW) capacity. In such situations, the retailer needs a rented warehouse (RW) to store the extra purchasing beyond his/her OW. Over purchasing also stimulates the deterioration if item is of perishable type. In general, most of the goods such as food stuffs, vegetables, and fruits having a time span to maintain the quality of freshness or original condition, i.e. no deterioration occurs during that span [9]. This property of a good is called non-instantaneous deterioration. In the literature, the planning horizon of such type of seasonal products is generally assumed as finite constant, but, in reality, due to the environmental effects, planning horizon fluctuates over years [24]. Hence, it is better to take planning horizon of such type of seasonal product as a random variable. Intuitively, in this paper we develop a two-warehouse inventory model for non-instantaneous deteriorating items in stochastic framework, wherein shortages are allowed and are mixture of partial backlog and lost sale. The brief review of literature related to the proposed work is presented below.

In the initial phase, [12] and [8] developed inventory models by considering deterioration. A comprehensive review of inventory models via deterioration is accomplished by [16]. We refer to that paper for exhaustive review. In recent years, many authors have developed deteriorating item inventory models for different types of business environments. [46] developed an EOQ model for deteriorating items, wherein time-varying demand and waiting time-dependent partial backlog have been considered. [30] developed an EOQ model for price-dependent demand rate and time-dependent deterioration rate – shortage are allowed and are partially backlogged. Inventory systems with deterioration of product via backlogging, partial backlogging, lost sale, inflation, delay in payments, and/ or varying demand are addressed by many authors such as [4,14,29,31,32,34–40].

We now briefly review the literature related to two-warehouse inventory modeling problem. [17] is the first who modeled inventory problem for limited storage capacity, and considered an additional storage facility with additional holding cost and called it as RW. [15] considered demand rate as a linear function of time

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for a two-level storage model, and included the cases of with and without shortage. In the last two-three decades, incorporation of deterioration in two-warehouse inventory modeling problems has received the attention of researches as well as enterprises. Perhaps, [41] first developed a two-warehouse inventory model in which deterioration of items and shortages are considered. [28] developed a two-warehouse inventory model, wherein demand rate was considered as stock-dependent. [49] considered complete backlog of shortages in a two-warehouse inventory system for deteriorating items, wherein demand is constant throughout the time horizon. [50] extended [49] by considering partial backlogging of the shortages. [20] extended two-warehouse deteriorating items inventory model by considering linear trend in demand rate, wherein shortages are allowed and are backlogged. [18] extended [20] by adding partial backlogging and inflation. [51] developed a production inventory model for deteriorating items, wherein production and demand rate are considered as time dependent. In recent years, many authors have developed two-warehouse inventory model for deteriorating items in different scenarios such as [1–3,27,33,42,43,45,47,52].

In the above discussion, we found that all the discussed models assumed a common characteristic, deterioration starts instantaneously just after their arrival in the stock. But, this assumption is not very realistic for the products which are mentioned in the first paragraph of this section. However, in recent years, many researchers addressed non-instantaneous deterioration in their models. [48] and [26] are among the first who incorporated non-instantaneous deterioration in deterministic inventory modeling problems. [6] enhanced the model of [48] by considering the objective of profit maximization, wherein shelf space is restricted. [11] extended [48] by considering time-dependent partial backlogging of shortages. [23] and [22] extended the model of [26] by incorporating price- and time-dependent demand rate and partial backlogging of shortages. [44] developed an inventory model for non-instantaneous time dependent deterioration and time-varying holding. Moreover, they assumed demand rate as a function of selling price and advertisement. Recently, [5] developed an inventory model for non-instantaneous deteriorating items, wherein various trade credits policies are discussed.

There is a common assumption in the above-discussed models, that is, all are developed in deterministic environment. However, some authors such as [7,10,19,21,25] have developed some models by considering stochastic review period. But, they have not considered two-warehouse or non-instantaneous deterioration rate. In this study we try to bridge this research gap. In this process we develop an inventory model, wherein a retailer has limited storage space and that is they use RW to keep the extra purchasing. Non-instantaneous deterioration that reflects the real world situation has been also incorporated. The model allows shortages which are partially backlogged and remaining are lost sales. In practice, it is rare that the planning horizon of the seasonal items is as constant. Thus, in this model we have considered the planning horizon as a random variable. To make the multifariousness in the model, we discuss two cases of distribution function of random time horizon, namely, uniform and truncated normal. The order-up-to-level (OU) replenishment system is the best when dealing with statistical uncertainty [10]. Hence, to ease in mathematical modeling of the proposed inventory problem, we follow the OU replenishment policy. To the best of our knowledge and as an evident of the above-discussed review of the literature as shown in Table 1, this problem is not deliberated by any one. Rest of the paper is organized as follows. Section 2 provides the list of notations and assumptions. Mathematical formulation of

Table 1. Contribution of different authors.

Author(s)	Deterioration	Shortage	Random time period time period	Two-warehouse
Sarma [41]	√	√		√
Goswami and Chaudhuri [15]		√		√
Kar et al. [20]	√	√		√
Ertogral and Rahim [10]		√	√	
Wee et al. [47]	√	√		√
Yang [49,50]	√	√		√
Chiang [7]		√	√	
Liu et al. [21]	√	√	√	
Sana [29]	√	√		
Jaggi et al. [18]	√	√		√
Sett et al. [42]	√			√
Panda et al. [27], and Agrawal et al. [1]	√			√
Cárdenas-Barrón et al. [4]	√			√
Karimi-Nasab and Konstantaras [19]		√	√	
Sarkar and Sarkar [37]	√			√
Tan and Weng [46]	√	√		
Taleizadeh and Nematollahi [45]	√			√
Yu et al. [52]	√			√
Bhunia et al. [2,3]				√
Shabani et al. [43]	√			√
Mohanty et al. [25]	√	√	√	
This paper	√	√	√	√

the model with solution methodology is presented in Section 3. Section 4 derives two special cases of random planning horizon, namely, uniform and truncated normal distribution. In Section 5, numerical examples with sensitivity analysis are presented in support of foregoing mathematical development and discussion. Finally, discussion is ended with Section 6 of concluding remarks and future extension of the research.

2. Notations and assumptions

The following notations to be used throughout the paper.

2.1. Notations

Parameters:

r	Demand rate per unit time
x	Random variable that represents the time/planning horizon($x_{\min} \le x \le x_{\max}$)
$f(x)$	Probability density function of x
$x_{\min}$	Minimum value of the random time horizon x
$x_{\max}$	Maximum value of the random time horizon x
t_p	Time period during which no deterioration
α	Deterioration rate at the RW
β	Deterioration rate at the OW
γ	Fraction of shortage quantity that is backlogged during shortage period ($0 < \gamma < 1$)
W	Capacity of OW
p	Purchasing price per unit the product
A	Ordering cost per order
h_r	Inventory holding cost per product per unit time at RW
h_o	Inventory holding cost per product per unit time at OW
b	Backlogging cost per product per unit time
l	Lost of sale cost per product

Decision variables:

S	Maximum inventory level

Dependant variables:

$i_r(t)$	Inventory level at the RW at time t
$i_o(t)$	Inventory level at the OW at time t
t_r	Time taken to consume the inventory of RW after the replenishment
t_o	Time taken to consume the inventory of OW after the replenishment
I_r	Accumulated inventory at RW
I_o	Accumulated inventory at OW
$\bar{D}$	Expected deteriorating items
$\bar{Q}$	Expected order quantity
$\bar{I}_r$	Expected inventory at RW
$\bar{I}_o$	Expected inventory at OW
$\bar{B}$	Expected shortage
$\bar{L}$	Expected lost sale quantity
ETC	Expected total cost

2.2. Assumptions

While the following assumptions are made in developing the model.

(1) Demand rate is uniform and constant throughout the horizon.
(2) Shortage is allowed, and is partially backlogged, i.e. a fraction γ ($0 \le \gamma \le 1$)of shortage quantity is backlogged and remaining is lost sale.
(3) The planning horizon is a random variable.
(4) The OW has a fixed capacity of W units, whereas the RW has no such limitation of stocking. The items of RW are consumed first and then of OW because inventory carrying cost at OW is higher than of RW.
(5) The OW is more improved than RW which prevents the product to deteriorate. Hence, the deterioration rate at the RW is higher than the deterioration rate at OW, i.e. $\alpha > \beta > 0$.
(6) Repairing or replacement for the deteriorated items are not permissible.

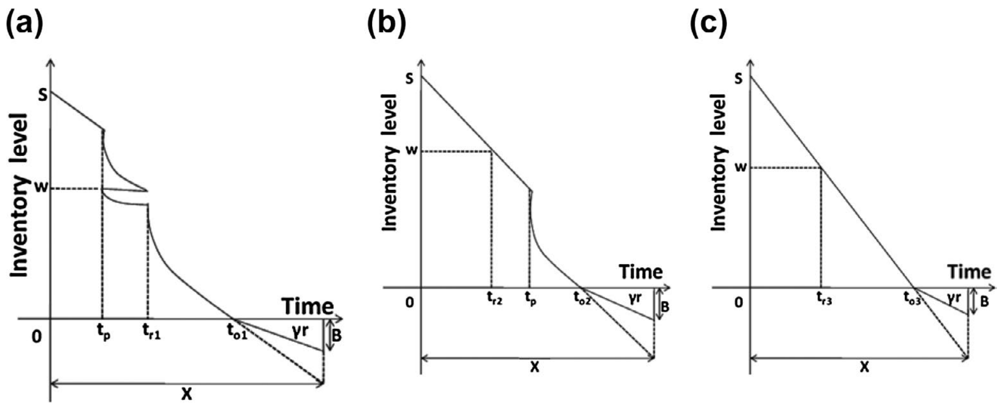

Figure 1. Inventory variation over random time horizon. (a) $t_p \le t_{r_1} \le t_{o_1}$, (b) $t_{r_2} \le t_p \le t_{o_2}$, and (c) $t_{r_3} \le t_{o_3} \le t_p$.

(7) The product has a fixed safe time period during which no deterioration occurs, i.e. t_p is a given constant.
(8) The distance between two-warehouses is known and transportation cost between them are constant. Hence, transportation cost and time from RW to OW are negligible.

3. Mathematical formulation of the model

In this section, we mathematically formulate the model and then analyze to find the global optimal solution. As illustrated in Figure 1, at the beginning of the cycle, a replenishment is occurred that brings the inventory level up to S. After that, inventory level is depleted due to demand or both demand and deterioration . Due to a fixed safe time t_p during which no deterioration occurs, and consumption time t_r and t_o of the products of RW and OW, three cases arise: (i) $t_p \le t_{r_1} \le t_{o_1}$, (ii) $t_{r_2} \le t_p \le t_{o_2}$, and (iii) $t_{r_3} \le t_{o_3} \le t_p$. In each case, RW inventory is first used to meet the demand of customer, and if it finished then OW inventory is used. When OW inventory reaches zero at t_{oj}, after that shortage occurs and is continued until the next replenishment. Now, each case is discussed in detail as follows.

Case (i). When $t_p \le t_{r_1} \le t_{o_1}$: As Figure 1(a) shows, inventory level at RW is depleted due to demand up to time t_p. After that deterioration occurs and inventory level of RW is depleted due to demand and deterioration up to time t_{r_1}, whereas OW inventory is depleted due to deterioration, only, during t_{r_1}. The RW inventory level t_{r_1} is governed by the following differential equations:

$$\begin{cases} \frac{di_{r_1}}{dt} = -r, & 0 \le t \le t_p \\ \frac{di_{r_1}}{dt} + \alpha i_{r_1} = -r, & t_p \le t \le t_{r_1} \end{cases} \tag{1}$$

with initial condition

$$i_{r_1}(0) = S - W \tag{2}$$

and boundary condition

$$i_{r_1}(t_{r_1}) = 0 \tag{3}$$

The OW is full with the maximum capacity W up to time t_p. During the period $[t_p, t_{r_1}] \cup [t_{r_1}, t_{o_1}]$ OW inventory level is governed by the following differential equations:

$$\begin{cases} \frac{di_{o_1}}{dt} + \beta i_{o_1} = 0, & t_p \le t \le t_{r_1} \\ \frac{di_{o_1}}{dt} + \beta i_{o_1} = -r, & t_{r_1} \le t \le t_{o_1} \end{cases} \tag{4}$$

with boundary condition

$$i_{o_1}(t_p) = W \text{ and } i_{o_1}(t_{o_1}) = 0 \tag{5}$$

After solving the Equations (1)–(5), the inventory levels of RW and OW can be described as follows:

$$i_{r_1}(t) = \begin{cases} -rt + S - W, & 0 \le t \le t_p \\ \frac{r}{\alpha}\{e^{-\alpha(t-t_{r_1})} - 1\}, & t_p \le t \le t_{r_1} \end{cases} \tag{6}$$

and

$$i_{o_1}(t) = \begin{cases} W, & 0 \le t \le t_p \\ We^{-\beta(t-t_p)}, & t_p \le t \le t_{r_1} \\ \frac{r}{\beta}\{e^{-\beta(t-t_{o_1})} - 1\}, & t_{r_1} \le t \le t_{o_1}. \end{cases} \tag{7}$$

As Figure 1(a) describes, inventory levels at both OW and RW are continuous functions of time. Continuity property of RW inventory level via Equation (6) gives

$$t_{r_1} = t_p + \frac{1}{\alpha}\ln\left[1 + \alpha\left(\frac{S-W}{r} - t_p\right)\right], \tag{8}$$

and continuity of OW inventory level via Equation (7) gives

$$t_{o_1} = t_{r_1} + \frac{1}{\beta}\ln\left[1 + \frac{W\beta}{r}\left(1 + \alpha\left(\frac{S-W}{r} - t_{r_1}\right)\right)^{-\frac{\beta}{\alpha}}\right] \tag{9}$$

As we discussed earlier, the review period is a random variable. So, for the fixed values of t_p, t_{r_1} and t_{o_1}, x may fall in any one of the intervals: (i) $[0, t_p]$, (ii) $[t_p, t_{r_1}]$, (iii) $[t_{r_1}, t_{o_1}]$, and (iv) $[t_{o_1}, \infty)$. Therefore, the total accumulated inventory at the RW and the OW, respectively, over the random interval $[0, x]$ are as follows:

$$I_{r_1}(x) = \begin{cases} \int_0^x i_{r_1}(t)dt, & 0 \le x \le t_p \\ \int_0^{t_p} i_{r_1}(t)dt + \int_{t_p}^x i_{r_1}(t)dt, & t_p \le x \le t_{r_1} \\ \int_0^{t_p} i_{r_1}(t)dt + \int_{t_p}^{t_{r_1}} i_{r_1}(t)dt, & t_{r_1} \le x \le t_{o_1} \\ \int_0^{t_p} i_{r_1}(t)dt + \int_{t_p}^{t_{r_1}} i_{r_1}(t)dt, & x \ge t_{o_1}. \end{cases} \tag{10}$$

and

$$I_{o_1}(x) = \begin{cases} \int_0^x i_{o_1}(t)dt, & 0 \le x \le t_p \\ \int_0^{t_p} i_{o_1}(t)dt + \int_{t_p}^x i_{o_1}(t)dt, & t_p \le x \le t_{r_1} \\ \int_0^{t_p} i_{o_1}(t)dt + \int_{t_p}^{t_{r_1}} i_{o_1}(t)dt + \int_{t_{r_1}}^x i_{o_1}(t)dt, & t_{r_1} \le x \le t_{o_1} \\ \int_0^{t_p} i_{o_1}(t)dt + \int_{t_p}^{t_{r_1}} i_{o_1}(t)dt + \int_{t_{r_1}}^{t_{o_1}} i_{o_1}(t)dt, & x \ge t_{o_1}. \end{cases} \tag{11}$$

The random planning horizon results random inventory level as described in Equations (10) and (11). Consequently, order quantity, backorder quantity, etc. are random variables. Thus, in order to scalarize these, we obtain the expected order quantity, expected inventory, expected deterioration, expected backorder, and lost sale, respectively, as follows:

$$\begin{aligned} \bar{Q}_1 = E(Q_1) &= \int_{x_{min}}^{t_p} rxf(x)dx \\ &+ \int_{t_p}^{t_{r_1}} (S - i_{r_1}(x) - i_{o_1}(x))f(x)dx + \int_{t_{r_1}}^{t_{o_1}} (S - i_{o_1}(x))f(x)dx \\ &+ \int_{t_{o_1}}^{x_{max}} [S + \gamma r(x - t_{o_1})]f(x)dx \end{aligned} \tag{12}$$

$$\bar{I}\bar{I}_{r_1} = E(I_{r_1}) = \int_{x_{\min}}^{t_p}\left(\int_0^x i_{r_1}(t)\mathrm{d}t\right)f(x)\mathrm{d}x + \int_{t_p}^{x_{\max}}\left(\int_0^{t_p} i_{r_1}(t)\mathrm{d}t\right)f(x)\mathrm{d}x + \int_{t_p}^{t_{r_1}}\left(\int_{t_p}^{x} i_{r_1}(t)\mathrm{d}t\right)f(x)\mathrm{d}x + \int_{t_{r_1}}^{x_{\max}}\left(\int_{t_p}^{t_{r_1}} i_{r_1}(t)\mathrm{d}t\right)f(x)\mathrm{d}x \quad (13)$$

$$\bar{I}\bar{I}_{o_1} = E(I_{o_1}) = \int_{x_{\min}}^{t_p}\left(\int_0^x i_{o_1}(t)\mathrm{d}t\right)f(x)\mathrm{d}x + \int_{t_p}^{x_{\max}}\left(\int_0^{t_p} i_{o_1}(t)dt\right)f(x)\mathrm{d}x + \int_{t_p}^{t_{r_1}}\left(\int_{t_p}^{x} i_{o_1}(t)dt\right)f(x)\mathrm{d}x + \int_{t_{r_1}}^{x_{\max}}\left(\int_{t_p}^{t_{r_1}} i_{o_1}(t)\mathrm{d}t\right)f(x)\mathrm{d}x + \int_{t_{r_1}}^{t_{o_1}}\left(\int_{t_{r_1}}^{x} i_{o_1}(t)\mathrm{d}t\right)f(x)\mathrm{d}x + \int_{t_{o_1}}^{x_{\max}}\left(\int_{t_{r_1}}^{t_{o_1}} i_{o_1}(t)\mathrm{d}t\right)f(x)\mathrm{d}x \quad (14)$$

$$\bar{D}_1 = E(D) = \int_{t_p}^{t_{r_1}}(S - i_r(x) - i_{o_1}(x) - rx)f(x)\mathrm{d}x + \int_{t_{r_1}}^{t_{o_1}}(S - i_{o1}(x) - rx)f(x)\mathrm{d}x + \int_{t_{o_1}}^{x_{\max}}[S - rt_{o_1}]f(x)\mathrm{d}x \quad (15)$$

$$\bar{B}_1 = E(B) = \int_{t_{o_1}}^{x_{\max}} \frac{1}{2}\gamma r(x - t_{o_1})^2 f(x)\mathrm{d}x \quad (16)$$

$$\bar{L}_1 = E(L) = \int_{t_{o_1}}^{x_{\max}} (1-\gamma)r(x - t_{o_1})f(x)\mathrm{d}x \quad (17)$$

The expected total cost is comprised with ordering, purchasing, holding, deterioration, backlogging costs, and lost sale. Let us denote expected total cost for this case as ETC_1.

$$ETC_1 = A + p\bar{Q}_1 + h_r\bar{I}_{r_1} + h_o\bar{I}_{o_1} + p\bar{D}_1 + b\bar{B}_1 + l\bar{L}_1 \quad (18)$$

Case (ii). When $t_{r_2} \le t_p \le t_{o_2}$: Inventory fluctuation at both warehouses in this case is shown in Figure 1(b). There is no deterioration in the RW, and inventory level is depleted due to demand and reached zero at the end of period t_{r_2}. After that OW inventory started to meet the demand. Up to t_p, OW inventory is depleted due to the demand, and then due to the demand and deterioration up to t_{o_2}. Then similar to Case (i), shortage occurs and is partially backlogged. The RW inventory for this case is governed by the following differential equations.

$$\frac{\mathrm{d}i_{r_2}}{\mathrm{d}t} = -r, \quad 0 \le t \le t_{r_2} \quad (19)$$

with initial condition

$$i_{r_2}(0) = S - W \quad (20)$$

and boundary condition

$$i_{r_2}(t_{r_2}) = 0. \quad (21)$$

The OW is full with the capacity W up to time t_{r_2}. Then inventory level during the period $[t_{r_2}, t_p] \cup [t_p, t_{o_2}]$ is governed by the following differential equations.

$$\begin{cases} \frac{\mathrm{d}i_{o_2}}{\mathrm{d}t} = -r, & t_{r_2} \le t \le t_p \\ \frac{\mathrm{d}i_{o_2}}{\mathrm{d}t} + \beta i_{o_2} = -r, & t_p \le t \le t_{o_2} \end{cases} \quad (22)$$

with boundary condition

$$i_{o_2}(t_{o_2}) = 0. \quad (23)$$

After solving the Equations (19)–(23), the inventory levels of RW and OW with respect to time are described as follows:

$$i_{r_2}(t) = -rt + S - W, \quad 0 \le t \le t_{r_2} \quad (24)$$

$$i_{o_2}(t) = \begin{cases} W, & 0 \le t \le t_{r_2} \\ -rt + S, & t_{r_2} \le t \le t_p \\ \frac{r}{\beta}\{e^{-\beta(t - t_{o_2})} - 1\}, & t_p \le t \le t_{o_2} \end{cases} \quad (25)$$

The boundary condition (21) gives

$$t_{r_2} = \frac{S - W}{r} \quad (26)$$

and the continuity of $i_{o_2}(t)$ at t_p gives

$$t_{o_2} = t_p + \frac{1}{\beta}\ln\left[1 + \beta\left(\frac{S}{r} - t_p\right)\right] \quad (27)$$

Similar to Case (i), for the fixed values of t_p, t_{r_2} and t_{o_2}, x may fall in any one of the intervals: (i) $[0, t_{r_2}]$, (ii) $[t_{r_2}, t_p]$, (iii) $[t_p, t_{o_2}]$, and (iv) $[t_{o_2}, \infty]$. Therefore, the total accumulated inventory over the random interval $[0, x]$ is:

$$I_{r_2}(x) = \begin{cases} \int_0^x i_{r_2}(t)\mathrm{d}t, & 0 \le x \le t_{r_2} \\ \int_0^{t_{r_2}} i_{r_2}(t)\mathrm{d}t, & t_{r_2} \le x \le t_p \\ \int_0^{t_{r_2}} i_{r_2}(t)\mathrm{d}t, & t_p \le x \le t_{o_2} \\ \int_0^{t_{r_2}} i_{r_2}(t)\mathrm{d}t, & x \ge t_{o_2}. \end{cases} \quad (28)$$

and

$$I_{o_2}(x)=\begin{cases}\int_0^x i_{o_2}(t)\mathrm{d}t, & 0\le x\le t_{r_2}\\ \int_0^{t_{r_2}} i_{o_2}(t)\mathrm{d}t+\int_{t_{r_2}}^{x} i_{o_2}(t)\mathrm{d}t, & t_{r_2}\le x\le t_p\\ \int_0^{t_{r_2}} i_{o_2}(t)\mathrm{d}t+\int_{t_{r_2}}^{t_p} i_{o_2}(t)\mathrm{d}t+\int_{t_p}^{x} i_{o_2}(t)\mathrm{d}t, & t_p\le x\le t_{o_2}\\ \int_0^{t_{r_2}} i_{o_2}(t)\mathrm{d}t+\int_{t_{r_2}}^{t_p} i_{o_2}(t)\mathrm{d}t+\int_{t_p}^{t_{o_2}} i_{o_2}(t)\mathrm{d}t, & x\ge t_{o_2}.\end{cases} \tag{29}$$

Consequently, the expected order quantity, expected inventory, expected deterioration, expected backorder, and lost sale, respectively, are as follows:

$$\bar{Q}_2=E(Q_2)=\int_{x_{\min}}^{t_p} rxf(x)\mathrm{d}x+\int_{t_p}^{t_{o_2}}(S-i_{o_2}(x))f(x)\mathrm{d}x+\int_{t_{o_2}}^{x_{\max}}[S+\gamma r(x-t_{o_2})]f(x)\mathrm{d}x \tag{30}$$

$$\bar{I}_{r_2}=E(I_{r_2})=\int_{x_{\min}}^{t_{r_2}}\left(\int_0^x i_{r_2}(t)\mathrm{d}t\right)f(x)\mathrm{d}x+\int_{t_{r_2}}^{x_{\max}}\left(\int_0^{t_{r_2}} i_{r_2}(t)\mathrm{d}t\right)f(x)\mathrm{d}x \tag{31}$$

$$\begin{aligned}\bar{I}_{o_2}=E(I_{o_2})=&\int_{x_{\min}}^{t_{r_2}}\left(\int_0^x i_{o_2}(t)\mathrm{d}t\right)f(x)\mathrm{d}x\\&+\int_{t_{r_2}}^{x_{\max}}\left(\int_0^{t_{r_2}} i_{o_2}(t)\mathrm{d}t\right)f(x)\mathrm{d}x+\int_{t_{r_2}}^{t_p}\left(\int_{t_{r_2}}^{x} i_{o_2}(t)\mathrm{d}t\right)f(x)\mathrm{d}x\\&+\int_{t_p}^{x_{\max}}\left(\int_{t_{r_2}}^{t_p} i_{o_2}(t)\mathrm{d}t\right)f(x)\mathrm{d}x+\int_{t_p}^{t_{o_2}}\left(\int_{t_p}^{x} i_{o_2}(t)\mathrm{d}t\right)f(x)\mathrm{d}x\\&+\int_{t_{o_2}}^{x_{\max}}\left(\int_{t_p}^{t_{o_2}} i_{o_2}(t)\mathrm{d}t\right)f(x)\mathrm{d}x\end{aligned} \tag{32}$$

$$\bar{D}_2=E(D_2)=\int_{t_p}^{t_{o_2}}(S-i_{o_2}(x)-rx)f(x)\mathrm{d}x+\int_{t_{o_2}}^{x_{\max}}[S-rt_{o_2}]f(x)\mathrm{d}x \tag{33}$$

$$\bar{B}_2=E(B_2)=\int_{t_{o_2}}^{x_{\max}}\frac{1}{2}\gamma r(x-t_{o_2})^2 f(x)\mathrm{d}x \tag{34}$$

$$\bar{L}_2=E(L_2)=\int_{t_{o_2}}^{x_{\max}}(1-\gamma)r(x-t_{o_2})f(x)\mathrm{d}x \tag{35}$$

Let us denote expected total cost for this case as ETC_2, then

$$ETC_2=A+p\bar{Q}_2+h_r\bar{I}_{r_2}+h_o\bar{I}_{o_2}+p\bar{D}_2+b\bar{B}_2+l\bar{L}_2 \tag{36}$$

Case (iii). When $t_{r_3}\le t_{o_3}\le t_p$: In this case, all items are consumed before to start the deterioration as shown in Figure 1(c). The RW inventory level is governed by the following differential equations.

$$\frac{\mathrm{d}i_{r_3}}{\mathrm{d}t}=-r,\quad 0\le t\le t_{r_3} \tag{37}$$

with initial condition

$$i_{r_3}(0)=S-W \tag{38}$$

and boundary condition

$$i_{r_3}(t_{r_3})=0 \tag{39}$$

The OW is full with the capacity W up to time t_{r_3}, and inventory level during the period $[t_{r_3}, t_{o_3}]$ is governed by the following differential equations.

$$i_{o_3}(t)=\frac{\mathrm{d}i_{o_3}}{\mathrm{d}t}=-r,\quad t_{r_3}\le t\le t_{o_3} \tag{40}$$

with boundary condition

$$i_{o_3}(t_{o_3})=0 \tag{41}$$

After solving the Equations (37)–(41). The RW and OW inventory levels can be expressed as follows.

$$i_{r_3}(t)=-rt+S-W,\quad 0\le t\le t_{r_3} \tag{42}$$

and

$$i_{o_3}(t)=\begin{cases}W, & 0\le t\le t_{r_3}\\ -rt+S, & t_{r_3}\le t\le t_{o_3}\end{cases} \tag{43}$$

The boundary conditions (39) and (41), respectively, give the following relationships.

$$t_{r_3}=\frac{S-W}{r} \tag{44}$$

$$t_{o_3}=\frac{S}{r}. \tag{45}$$

Similar to Case (i) and Case (ii), for the fixed values of t_p, t_{r_3} and t_{o_3}, x may fall in any one of the intervals: (i) $[0\ t_{r_3}]$, (ii) $[t_{r_3}, t_{o_3}]$, (iii) $[t_{o_3}, t_p]$, and (iv) $[t_p, \infty)$. Therefore, the total accumulated inventory at RW and OW, respectively, over the random interval $[0, x]$ are:

$$I_{r_3}(x)=\begin{cases}\int_0^x i_{r_3}(t)\mathrm{d}t, & 0\le x\le t_{r_3}\\ \int_0^{t_{r_3}} i_{r_3}(t)\mathrm{d}t, & x\ge t_{r_3}.\end{cases} \tag{46}$$

$$I_{o_3}(x) = \begin{cases} \int_0^x i_{o_3}(t)\mathrm{d}t, & 0 \le x \le t_{r_3} \\ \int_0^{t_{r_3}} i_{o_3}(t)\mathrm{d}t + \int_{t_{r_3}}^{x} i_{o_3}(t)\mathrm{d}t, & t_{r_3} \le x \le t_{o_3} \\ \int_0^{t_{r_3}} i_{o_3}(t)\mathrm{d}t + \int_{t_{r_3}}^{t_{o_3}} i_{o_3}(t)\mathrm{d}t, & x \ge t_{o_3}. \end{cases} \quad (47)$$

Consequently, the expected order quantity, expected inventory, expected deterioration, expected back-order, and lost sale, respectively, are as follows:

$$\bar{Q}_3 = E(Q_3) = \int_{x_{\min}}^{t_{o_3}} rxf(x)\mathrm{d}x + \int_{t_{o_3}}^{x_{\max}} [S + \gamma r(x - t_{o_3})]f(x)\mathrm{d}x \quad (48)$$

$$\bar{I}_{r_3} = E(I_{r_3}) = \int_{x_{\min}}^{t_{r_3}} \left(\int_0^x i_{r_3}(t)\mathrm{d}t\right)f(x)\mathrm{d}x + \int_{t_{r_3}}^{x_{\max}} \left(\int_0^{t_{r_3}} i_{r_3}(t)\mathrm{d}t\right)f(x)\mathrm{d}x \quad (49)$$

$$\bar{I}_{o_3} = E(I_{o_3}) = \int_{x_{\min}}^{t_{r_3}} \left(\int_0^x i_{o_3}(t)\mathrm{d}t\right)f(x)\mathrm{d}x + \int_{t_{r_3}}^{x_{\max}} \left(\int_0^{t_{r_3}} i_{o_3}(t)\mathrm{d}t\right)f(x)\mathrm{d}x + \int_{t_{r_3}}^{t_{o_3}} \left(\int_{t_{r_3}}^{x} i_{o_3}(t)\mathrm{d}t\right)f(x)\mathrm{d}x + \int_{t_{o_3}}^{x_{\max}} \left(\int_{t_{r_3}}^{t_{o_3}} i_{o_3}(t)\mathrm{d}t\right)f(x)\mathrm{d}x \quad (50)$$

$$\bar{B}_3 = E(B_3) = \int_{t_{o_3}}^{x_{\max}} \frac{1}{2}\gamma r(x - t_{o_3})^2 f(x)\mathrm{d}x \quad (51)$$

$$\bar{L}_3 = E(L_3) = \int_{t_{o_3}}^{x_{\max}} (1-\gamma)r(x - t_{o_3})f(x)\mathrm{d}x \quad (52)$$

Let us denote expected total cost for this case as ETC_3. Hence,

$$ETC_3 = A + p\bar{Q}_3 + h_r\bar{I}_{r_3} + h_o\bar{I}_{o_3} + b\bar{B}_3 + l\bar{L}_3 \quad (53)$$

We have discussed three cases depending upon the values of t_p, t_{rj}, and t_{oj}. We now combine all three cases to find the integrated expected total cost ETC, as:

$$ETC = \begin{cases} ETC_1, & t_p \le t_{r_1} \le t_{o_1}; \\ ETC_2, & t_{r_2} \le t_p \le t_{o_2}; \\ ETC_3, & t_{r_3} \le t_{o_3} \le t_p. \end{cases} \quad (54)$$

It is not possible to optimize analytically the cost function (54) via ETC_1, ETC_2 and ETC_3 using any optimization method. Thus, we use Mathematica software to optimize the integrated cost function. However, the necessary and sufficient conditions of minimizing the cost function are

$$\frac{\mathrm{d}(ETC_j)}{\mathrm{d}S} = 0 \quad \text{and} \quad \left.\frac{\mathrm{d}^2(ETC_j)}{\mathrm{d}S^2}\right|_{S=S_j^*} > 0 \quad \text{for} \quad j = 1,2,3 \quad (55)$$

where S_j^* is an extremum point obtained from the equation $\mathrm{d}(ETC_j)/\mathrm{d}S = 0$. The convexity of cost function is shown in Figures 2–5. We here write the solution procedure in order to find the global optimal solution.

Step 1. Minimize ETC_1 subject to the constraint $t_{r_1} \ge t_p$ using the command NMinimize of Mathematica.

Step 2. Minimize ETC_2 subject to the constraints $t_{r_2} \le t_p$ and $t_p \le t_{o_2}$ using the command NMinimize of Mathematica.

Step 3. Minimize ETC_3 subject to the constraint $t_{o_3} \le t_p$ using the command NMinimize of Mathematica.

Step 4. Set, $j^* = \arg(\min_{j\in\{1,2,3\}} ETC_j^*)$ where ETC_{j^*} is the minimum value of ETC_j^*.

Step 5. Optimal solution is, $ETC^* = ETC_{j^*}$, $S^* = S_{j^*}$, $t_r^* = t_{rj^*}$, and $t_o^* = t_{oj^*}$.

4. Special cases

In this section, two cases of random planning horizon, namely, uniform and truncated normal distributions are considered.

4.1. Uniform distribution function

In many cases, data are available with incomplete knowledge about the distribution type. In such a case, it is difficult to fit any distribution other than uniform. This situation can be contemplated by considering random time horizon that is uniformly distributed as

$f(x) = 1/(x_{\max} - x_{\min})$, $x_{\min} \le x \le x_{\max}$. This assumption also captures the situation that random time horizon is equally likely in the interval $[x_{\min}, x_{\max}]$. However, the expected order quantity, inventory level, etc. for all three cases are obtained in Appendix 1.

4.2. Truncated normal distribution function

According to the central limit theorem, for a sufficiently large number of any independent and identically distributed random variables, the approximated distribution is normal. Moreover, the normal distribution is easier to use in calculation point of view. Hence, it is one of the most useful distributions. But, in dealing with normal distribution, an unrealistic situation incur regarding its range that is $-\infty$ to ∞. It is impossible to take time horizon as any negative real number or ∞. The general idea about area of normal curve is that the interval $[\rho - 3\sigma, \rho + 3\sigma]$ cover 99.73% of total area, where ρ is mean and σ is standard deviation. Hence, for convenience we truncated normal distribution to an interval $[x_{\min}, x_{\max}]$, where $0 \le x_{\min} \le \rho - 3\sigma$ and $x_{\max} \ge \rho + 3\sigma$. Thus, time horizon is truncated normally distributed as, $f(x) = \frac{1}{\Phi(x_{\max})-\Phi(x_{\min})} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}(\frac{x-\rho}{\sigma})^2}, x_{\min} \le x \le x_{\max}$, where Φ is c.d.f. of standard normal distribution. For this distribution function, the expected values of order quantity, inventory level, etc. are obtained in Appendix 2.

5. Numerical experiment with sensitivity analysis

In this section, forgoing discussion is elaborated with some numerical examples. The data are taken from an article by [13] after some modification as per our model requirement.

5.1. Uniform distribution function

Example 1. Let us consider an inventory control problem with the following input data: demand rate, $r = 10$ units/month, OW capacity, $W = 25$ units, deterioration rate at RW, $\alpha = 0.01$, deterioration rate at OW, $\beta = 0.02$, backlogging fraction, $\gamma = 0.5$, purchasing price, $p = \$5$/unit, holding cost at RW, $h_r = \$0.2$/unit/month, holding cost at OW, $h_o = \$0.1$/unit/month, ordering cost, $A = \$100$ per order, backlogging cost $b = \$2$/unit/month, lost sales, and $l = \$10$/unit item. In order to examine the effectiveness of the model, we take two values of t_p as, $t_p = 2$ months and $t_p = 5$ months, and two instances of uniform distribution parameters are taken as, (i) $x_{\min} = 1$ month, $x_{\max} = 5$ months and (ii) $x_{\min} = 3$ months, $x_{\max} = 8$ months. Convexity of the cost function for this data-set is shown in Figures 2 and 3 which ensure uniqueness of optimal solution.

For instance (i) with $t_p = 2$ months, Figure 2(a) shows the uniqueness of optimal solution, and is: $S^* = 41.3175$ units and $ETC^* = \$264.017$, which is optimized by the cost function ETC_2 of the integrated cost function ETC. Figure 2(b) shows the convexity of cost function for instance (i) with $t_p = 5$ months, the optimal solution is: $S^* = 43.3686$ units and $ETC^* = \$261.014$ which is obtained from third case. For instance (ii) with $t_p = 2$ months, as Figure 3(a) shows, ETC_1 optimizes the ETC with optimal solution $S^* = 61.4417$ units and $ETC^* = \$425.785$, and for $t_p = 5$ months, ETC_2 (Figure 3(b)) optimizes the ETC to $S^* = 64.1208$ units and $ETC^* = \$410.69$ for this numerical example.

5.1.1. Sensitivity of demand and deterioration

"How the changing demand rate and deterioration rate make effects on the decision policy?" are examined here. For this we vary the demand rate and deterioration rate singly from −40% to 40%. The performances of changing parameters are obtained in Tables 2–5 for instance (i) and

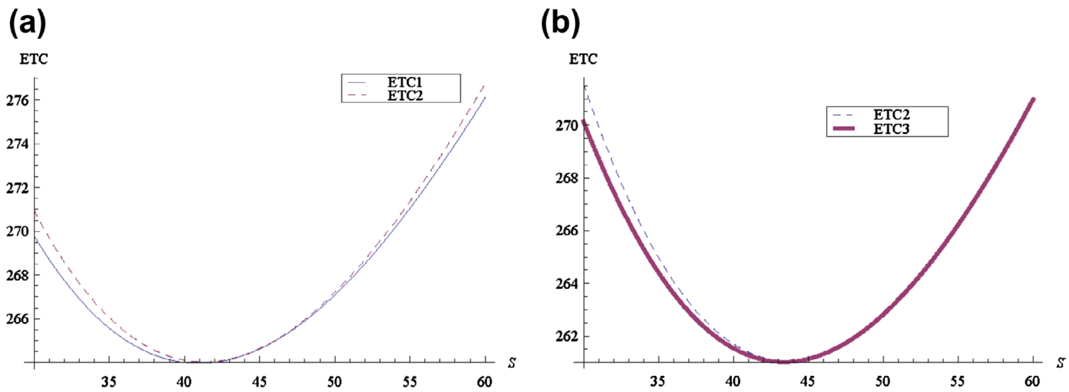

Figure 2. Convexity of instance (i) for uniformly distributed time horizon. (a) *tp* = 2, and (b) *tp* = 5.

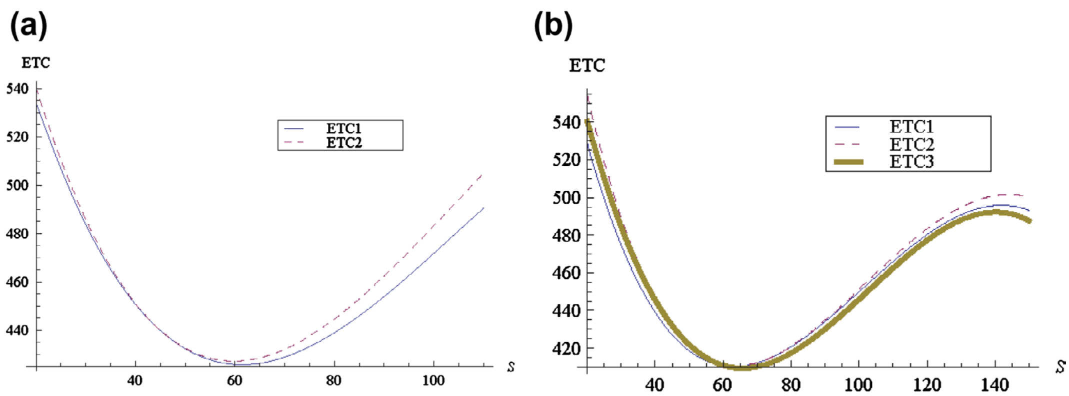

Figure 3. Convexity of instance (ii) for uniformly distributed time horizon. (a) *tp* = 2, and (b) *tp* = 5

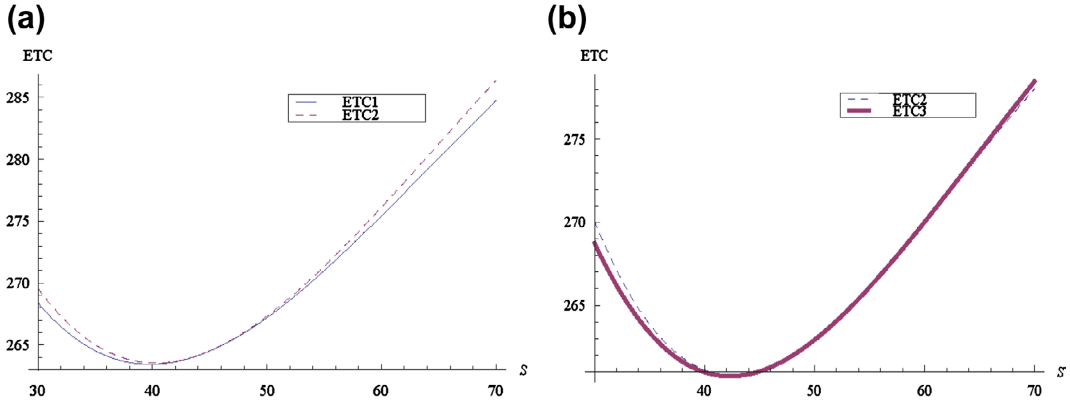

Figure 4. Convexity of instance (i) for truncated normal distributed time horizon. (a) $tp = 2$, and (b) $tp = 5$.

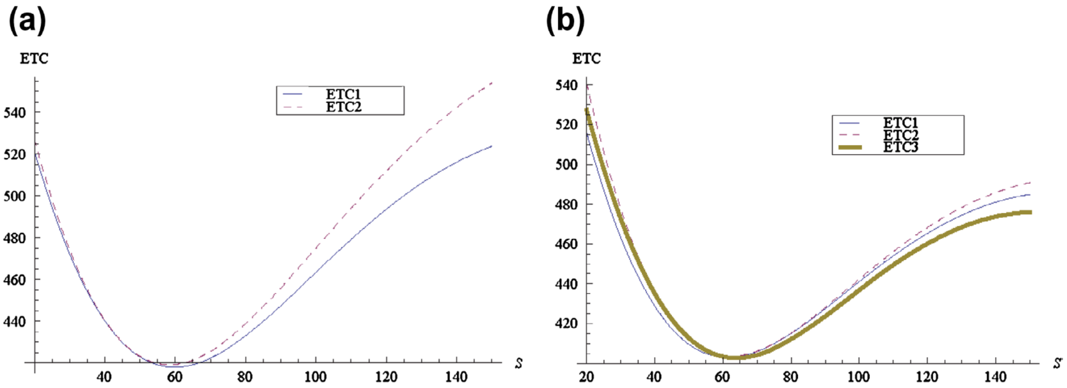

Figure 5. Convexity of instance (ii) for truncated normal distributed time horizon. (a) $tp = 2$, and (b) $tp = 5$.

Table 2. Sensitivity of demand for uniformly distributed time horizon with $t_p = 2$.

x_{min}	x_{max}	% Change	r	S^*	j^*	$\bar{Q}^*$	$\bar{D}^*$	$\bar{B}^*$	$\bar{L}^*$	t_r^*	t_o^*	ETC^*
1	5	−40	6	25.8925	2	17.9719	0.175414	0.049957	0.203465	0.14875	4.2634	197.55
		−20	8	33.5387	2	23.8487	0.213654	0.103937	0.364965	1.06734	4.14564	230.548
		0	10	41.3175	2	29.7345	0.254829	0.158263	0.520348	1.63175	4.08755	264.017
		20	12	49.1754	1	35.6281	0.297674	0.210889	0.669583	2.01462	4.05513	297.688
		40	14	57.3841	1	41.5638	0.342719	0.244988	0.778957	2.31266	4.05648	331.415
3	8	−40	6	36.8909	2	32.6986	0.91625	0.817713	1.21765	1.98182	5.98535	293.616
		−20	8	49.1764	1	43.5583	1.17623	1.08461	1.6179	3.01686	5.98885	359.717
		0	10	61.4417	1	54.3729	1.38521	1.34568	2.01233	3.6308	5.99385	425.785
		20	12	73.6882	1	65.1664	1.57197	1.60556	2.40556	4.03647	5.99769	491.83
		40	14	85.9236	1	75.9492	1.74736	1.86487	2.7982	4.32446	6.00064	557.862

Table 3. Sensitivity of demand for uniformly distributed time horizon with $t_p = 5$.

x_{min}	x_{max}	% Change	r	S^*	j^*	$\bar{Q}^*$	$\bar{D}^*$	$\bar{B}^*$	$\bar{L}^*$	t_r^*	t_o^*	ETC^*
1	5	−40	6	27.1308	3	17.9142	0	0.0136688	0.0857521	0.355136	4.5218	195.543
		−20	8	35.1814	3	23.8186	0	0.0364202	0.181398	1.27267	4.39767	228.055
		0	10	43.3686	3	29.7252	0	0.0607549	0.27485	1.83686	4.33686	261.014
		20	12	51.6232	3	35.6345	0	0.0850426	0.365476	2.2186	4.30193	294.164
		40	14	59.9162	3	41.5461	0	0.108986	0.453938	2.49402	4.27973	327.411
3	8	−40	6	39.4792	2	32.4468	0.0728365	0.301453	0.626038	2.41321	6.55543	283.277
		−20	8	51.7485	2	43.121	0.0852466	0.499014	0.96422	3.34356	6.44741	346.759
		0	10	64.1208	2	53.8073	0.0993047	0.692305	1.29202	3.91208	6.39251	410.69
		20	12	76.5437	2	64.5003	0.114122	0.882227	1.61381	4.29531	6.35998	474.818
		40	14	88.9953	2	75.1975	0.129353	1.06979	1.93188	4.57109	6.33873	539.049

instance (ii). Tables 2 and 3 show that the performances of changing demand rate for both instances. This evinces when demand rate is increased, order up to level (S^*) increases, consequently, expected total cost increases. This is a natural conscious when demand rate increases, decision-maker has to procure more quantity, consequently, increment in S^* is noticed. Demand rate also affects the consumption time of OW and RW inventories. When demand rate is increased then for both instances, t_r^* increased, whereas t_o^* decreased. The increment in t_r^* is an outcome of joint effect of more stocking in RW and increasing demand rate.

Decrement in t_o^* is an outcome of increasing demand rate while fixed capacity of OW. The effectiveness of deterioration rate is shown in Tables 4 and 5. When it is increased, S^* slightly decreases, whereas ETC increases slightly. Due to increasing deterioration rate, comparatively lesser amount has been kept in the stock. Total cost is increased because more items have been deteriorated and that's why deterioration cost has been increased. As Tables 2–5 delineate demand rate is more sensitive than deterioration rate. When t_p is increased from 2 to 5, then order quantities almost same whereas order-up-to levels slightly increase. It is because when deterioration starts later then comparatively fewer items will deteriorate. Table 5 shows that for the case $x_{min} = 1$ and $x_{max} = 5$, the optimal solution is remains same for different values of α and β because in this case no item is deteriorated. Hence, more stocking increases the order-up-to level and decreases backorder cost and lost sale, which finally results decrement in the total cost.

5.2. Truncated normal distribution function

Example 2. We now suppose that the time horizon is truncated in the interval $[x_{min}, x_{max}]$ for normally distributed random variable. The value of the other parameters is as same as Example 1. Moreover, we consider two instances of truncated normal distribution as, (i) $x_{min} = 1$ month, $x_{max} = 5$, $\rho = 3$, $\sigma = 2$ and (ii) $x_{min} = 3$ months, $x_{max} = 8$ months, $\rho = 5$, $\sigma = 3$. Convexity of cost function for this data-set is shown in Figures 4 and 5. The optimum solution for both instances is obtained in Tables 6–9. For instance (i) with $t_p = 2$ months, as Figure 4(a) shows, ETC_2 optimizes the ETC to the global optimum as $S^* = 40.3031$ units and $ETC^* = \$263.547$. On the other hand, for same instance with $t_p = 5$ months, optimum solution is obtained from case (iii) (shown in Figure 4(b)) as $S^* = 42.3286$ units and $ETC^* = \$260.733$. Furthermore, for instance (ii) with $t_p = 2$ months, the optimal solution is $S^* = 59.8088$ units and $ETC^* = \$417.985$; and with $t_p = 5$ months, the optimal solution is $S^* = 62.5915$ units and $ETC^* = \$403.638$. Convexity of the cost function for instance (ii) is shown in Figures 5(a) and (b), respectively.

Table 4. Sensitivity of deterioration for uniformly distributed time horizon with $t_p = 2$.

x_{min}	x_{max}	% Change	α	β	S^*	j^*	$\bar{Q}^*$	$\bar{D}^*$	$\bar{B}^*$	$\bar{L}^*$	t_r^*	t_o^*	ETC^*
1	5	−40	0.006	0.012	42.1143	2	29.7463	0.163923	0.113782	0.417595	1.71143	4.18259	262.875
		−20	0.008	0.016	41.7116	2	29.7427	0.21108	0.135163	0.468396	1.67116	4.1343	263.455
		0	0.01	0.02	41.3175	2	29.7345	0.254829	0.158263	0.520348	1.63175	4.08755	264.017
		20	0.012	0.024	40.9321	2	29.7222	0.295367	0.182983	0.573212	1.59321	4.04233	264.56
		40	0.014	0.028	40.5557	2	29.7061	0.332887	0.209218	0.626768	1.55557	3.99859	265.085
3	8	−40	0.006	0.012	63.1155	1	54.294	0.890517	0.950975	1.59655	3.80177	6.21307	419.55
		−20	0.008	0.016	62.2769	1	54.3462	1.14737	1.13951	1.80115	3.71586	6.10203	422.72
		0	0.01	0.02	61.4417	1	54.3729	1.38521	1.34568	2.01233	3.6308	5.99385	425.785
		20	0.012	0.024	60.6108	1	54.3754	1.60466	1.56904	2.22926	3.54664	5.88848	428.746
		40	0.014	0.028	59.785	1	54.3552	1.80636	1.80911	2.45121	3.4634	5.78586	431.607

Table 5. Sensitivity of deterioration for uniformly distributed time horizon with $t_p = 5$.

x_{min}	x_{max}	% Change	α	β	S^*	j^*	$\bar{Q}^*$	$\bar{D}^*$	$\bar{B}^*$	$\bar{L}^*$	t_r^*	t_o^*	ETC^*
1	5	40	0.014	0.028	43.3686	3	29.7252	0	0.0607549	0.27485	1.83686	4.33686	261.014
		20	0.012	0.024	43.3686	3	29.7252	0	0.0607549	0.27485	1.83686	4.33686	261.014
		0	0.01	0.02	43.3686	3	29.7252	0	0.0607549	0.27485	1.83686	4.33686	261.014
		−20	0.008	0.016	43.3686	3	29.7252	0	0.0607549	0.27485	1.83686	4.33686	261.014
		−40	0.006	0.012	43.3686	3	29.7252	0	0.0607549	0.27485	1.83686	4.33686	261.014
3	8	−40	0.006	0.012	64.6983	2	53.8741	0.0644454	0.61225	1.19039	3.96983	6.45702	410.202
		−20	0.008	0.016	64.4049	2	53.8409	0.0826029	0.652233	1.24167	3.94049	6.42414	410.451
		0	0.01	0.02	64.1208	2	53.8073	0.0993047	0.692305	1.29202	3.91208	6.39251	410.69
		20	0.012	0.024	63.8457	2	53.7732	0.114662	0.732381	1.34141	3.88457	6.36207	410.92
		40	0.014	0.028	63.5795	2	53.739	0.128778	0.772387	1.38983	3.85795	6.33277	411.142

Table 6. Sensitivity of demand for truncated normally distributed time horizon with $t_p = 2$.

x_{min}	x_{max}	% Change	r	S^*	j^*	$\bar{Q}^*$	$\bar{D}^*$	$\bar{B}^*$	$\bar{L}^*$	t_r^*	t_o^*	ETC^*
1	5	−40	6	25.2825	2	17.9556	0.165677	0.0566651	0.210051	0.0470835	4.16614	197.363
		−20	8	32.7407	2	23.8321	0.201414	0.113142	0.369334	0.967582	4.04998	230.218
		0	10	40.3031	2	29.7137	0.239462	0.17093	0.525773	1.53031	3.99018	263.547
		20	12	47.9257	2	35.6001	0.278785	0.22809	0.67865	1.91047	3.95508	297.09
		40	14	55.8122	1	41.5185	0.321181	0.271486	0.802724	2.20067	3.94833	330.723
3	8	−40	6	36.0167	2	32.0589	0.853936	0.797387	1.16334	1.83612	5.85064	288.92
		−20	8	47.8658	1	42.6816	1.09805	1.08617	1.57418	2.85456	5.83592	353.463
		0	10	59.8088	1	53.2862	1.29777	1.34873	1.95873	3.47002	5.8405	417.985
		20	12	71.7331	1	63.8688	1.47429	1.61009	2.34208	3.8767	5.84407	482.486
		40	14	83.6463	1	74.4402	1.6389	1.87088	2.72483	4.16541	5.84684	546.974

Table 7. Sensitivity demand for truncated normally distributed time horizon with $t_p = 5$.

x_{min}	x_{max}	% Change	r	S^*	j^*	$\bar{Q}^*$	$\bar{D}^*$	$\bar{B}^*$	$\bar{L}^*$	t_r^*	t_o^*	ETC^*
1	5	−40	6	26.5842	3	17.9057	0	0.0175071	0.0942639	0.264026	4.43069	195.463
		−20	8	34.3932	3	23.8058	0	0.0442052	0.194173	1.17415	4.29915	227.872
		0	10	42.3286	3	29.7064	0	0.0730024	0.293558	1.73286	4.23286	260.733
		20	12	50.3346	3	35.6096	0	0.10182	0.390419	2.11122	4.19455	293.797
		40	14	58.3838	3	41.515	0	0.130206	0.485001	2.38456	4.17027	326.964
3	8	−40	6	38.5909	2	31.8186	0.0573132	0.310016	0.607054	2.26515	6.4117	279.192
		−20	8	50.541	2	42.2882	0.0659134	0.510223	0.935463	3.19262	6.30056	341.185
		0	10	62.5915	2	52.7681	0.0759481	0.706855	1.25505	3.75915	6.24356	403.638
		20	12	74.6953	2	63.2545	0.0866808	0.900048	1.5688	4.14127	6.20985	466.296
		40	14	86.8303	2	73.7452	0.0978031	1.09069	1.87873	4.41645	6.18794	529.062

Table 8. Sensitivity of deterioration for truncated normally distributed time horizon with $t_p = 2$.

x_{min}	x_{max}	% Change	α	β	S^*	j^*	$\bar{Q}^*$	$\bar{D}^*$	$\bar{B}^*$	$\bar{L}^*$	t_r^*	t_o^*	ETC^*
1	5	−40	0.006	0.012	41.0775	2	29.7244	0.154062	0.127366	0.429678	1.60775	4.08154	262.478
		−20	0.008	0.016	40.6846	2	29.721	0.198344	0.148475	0.477329	1.56846	4.03497	263.022
		0	0.01	0.02	40.3031	2	29.7137	0.239462	0.17093	0.525773	1.53031	3.99018	263.547
		20	0.012	0.024	39.9329	2	29.7028	0.277622	0.194637	0.574833	1.49329	3.94708	264.056
		40	0.014	0.028	39.5735	2	29.6887	0.313016	0.219504	0.62435	1.45735	3.90559	264.548
3	8	−40	0.006	0.012	61.4218	1	53.2108	0.834593	0.97941	1.57104	3.63414	6.05023	412.154
		−20	0.008	0.016	60.6114	1	53.2604	1.07508	1.15666	1.76187	3.55147	5.9438	415.119
		0	0.01	0.02	59.8088	1	53.2862	1.29777	1.34873	1.95873	3.47002	5.8405	417.985
		20	0.012	0.024	59.014	1	53.2895	1.50329	1.55528	2.161	3.38975	5.74018	420.755
		40	0.014	0.028	58.227	1	53.2713	1.69225	1.77595	2.36811	3.3106	5.6427	423.43

Table 9. Sensitivity of deterioration for truncated normally distributed time horizon with $t_p = 5$.

x_{min}	x_{max}	% Change	α	β	S^*	j^*	$\bar{Q}^*$	$\bar{D}^*$	$\bar{B}^*$	$\bar{L}^*$	t_r^*	t_o^*	ETC^*
1	5	40	0.014	0.028	42.3286	3	29.7064	0	0.0730024	0.293558	1.73286	4.23286	260.733
		20	0.012	0.024	42.3286	3	29.7064	0	0.0730024	0.293558	1.73286	4.23286	260.733
		0	0.01	0.02	42.3286	3	29.7064	0	0.0730024	0.293558	1.73286	4.23286	260.733
		−20	0.008	0.016	42.3286	3	29.7064	0	0.0730024	0.293558	1.73286	4.23286	260.733
		−40	0.006	0.012	42.3286	3	29.7064	0	0.0730024	0.293558	1.73286	4.23286	260.733
3	8	−40	0.006	0.012	63.0885	2	52.8242	0.0491259	0.639982	1.17211	3.80885	6.29868	403.264
		−20	0.008	0.016	62.8359	2	52.7963	0.0630699	0.673493	1.21399	3.78359	6.27059	403.455
		0	0.01	0.02	62.5915	2	52.7681	0.0759481	0.706855	1.25505	3.75915	6.24356	403.638
		20	0.012	0.024	62.3551	2	52.7398	0.0878404	0.740024	1.29529	3.73551	6.21755	403.815
		40	0.014	0.028	62.1263	2	52.7113	0.0988206	0.772961	1.3347	3.71263	6.1925	403.985

5.2.1. Sensitivity of demand and deterioration

Similar to Example 1, we vary the demand rate and deterioration rate from −40% to 40% in order to find the effective of demand and deterioration in case of truncated normal distributed time horizon. The sensitivity of changing demand rate is shown in Tables 6 and 7, and sensitivity of deterioration is shown in Tables 8 and 9. These tables evince that the tendencies of effectiveness of the parameters are same as of the Example 1. However, a little difference in the magnitude of the decision variable obtained in both examples has been found. Similar to Example 1, when t_p is increased from 2 to 5, then order quantities are almost the same, whereas order-up-to levels slightly increase, as shown in Tables 6–9.

Throughout our investigation we found that demand rate is more sensitive than deterioration rate. Furthermore, truncated normal distribution over performs to the uniform distribution.

6. Conclusion

In this study, we have presented a single-period integrated inventory model that simultaneously deals with non-instantaneous deterioration of item, two-warehouse storing facility, and partial backlogging of shortages in stochastic environment. To capture the real life business situation, we have considered that time horizon as a random variable and focused on two distributions of random time horizon (1) uniform and (2) truncated normal. In this paper, first time, two-warehouse inventory model is developed in stochastic environment for non-instantaneous deteriorating item. In real life, most of the seasonal commodities such as food stuff, vegetables, and fruits have a time span to maintain their freshness or original condition. Moreover, such type of products have a particular season which may vary randomly over the seasons. This model can be applied while dealing with such type of products under above discussed scenarios.

The competency of the model is elaborated with some illustrative examples and sensitivity analysis for changing values of parameters. Throughout in our investigation we have found that truncated normally distributed time horizon over performs to the uniform distribution. Moreover, for the both distributions, demand rate highly influences the decision-making policy.

Some potential extensions of the model are: (1) Time horizon can be considered as infinite with stochastic review period; (2) Preservation technology can be included to prevent the deterioration; (3) This model can be extended for non-constant, non-deterministic demand rate over stochastic time horizon.

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No potential conflict of interest was reported by the authors.

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Appendix 1.

Case (i) $t_p \le t_{r_1} \le t_{o_1}$

$$\bar{Q}_1 = \frac{1}{x_{\max} - x_{\min}} \left\{ \begin{array}{l} \frac{r}{\beta}\left(t_{o_1} - t_{r_1}\right) + \frac{r}{\alpha} t_{r_1} \\ +S\left(x_{\max} - t_p\right) - \frac{S-W}{\alpha} + \frac{\gamma r}{2}\left(x_{\max} - t_{o_1}\right)^2 \end{array} \right.$$

$$\left. + \frac{r}{2}\left(t_p^2 - x_{\min}^2\right) - \frac{W}{\beta} \right\}$$

$$\bar{I}_{r_1} = \frac{1}{x_{\max} - x_{\min}} \left\{ \begin{array}{l} -\frac{r}{2\alpha}\left(t_{r_1} - t_p\right)^2 \\ -\frac{r}{\alpha^2}\left(\alpha t_{r_1} - \alpha t_p + 1\right)\left(x_{\max} - t_{r_1}\right) \\ -\frac{S-W}{\alpha^2} + \frac{S-W}{2}\left(t_p^2 - x_{\min}^2\right) \end{array} \right.$$

$$\left. \begin{array}{l} +r\left(\frac{1}{\alpha^2} + \frac{1}{\alpha}\left(\frac{S-W}{r} - t_p\right) + \frac{S-W}{r} t_p - \frac{t_p^2}{2}\right)\left(x_{\max} - t_p\right) \\ -\frac{r}{6}\left(t_p^3 - x_{\min}^3\right) + \frac{r}{\alpha^2} t_p \end{array} \right\}$$

$$\bar{I}_{o_1} = \frac{1}{x_{\max} - x_{\min}} \left\{ \begin{array}{l} -\frac{r}{2\beta}\left(t_{o_1} - t_{r_1}\right)^2 + \frac{r}{\beta^2}\left(t_{o_1} - t_{r_1} - 1\right) \\ -\frac{r}{\beta}\left(t_{o_1} - t_{r_1}\right)\left(x_{\max} - t_{o_1}\right) + \frac{W}{2}\left(t_p^2 - x_{\min}^2\right) \end{array} \right.$$

$$\left. + \frac{W}{\beta}\left(1 + \beta t_p\right)\left(x_{\max} - t_p\right) \right\}$$

$$\bar{D}_1 = \frac{1}{x_{\max} - x_{\min}} \left\{ \begin{array}{l} \frac{r}{2}\left(t_{o_1}^2 + t_p^2\right) - r t_{o_1} x_{\max} \\ +\frac{r}{\alpha} t_{r_1} - \frac{S-W}{\alpha} + S\left(x_{\max} - t_p\right) \end{array} \right\}$$

$$\bar{B}_1 = \frac{1}{x_{\max} - x_{\min}} \frac{\gamma r}{6}\left(x_{\max} - t_{o_1}\right)^3$$

$$\bar{L}_1 = \frac{1}{x_{\max} - x_{\min}} \frac{r}{2}(1-\gamma)\left(x_{\max} - t_{o_1}\right)^2$$

Case (ii) $t_{r_2} \le t_p \le t_{o_2}$

$$\bar{Q}_2 = \frac{1}{x_{\max} - x_{\min}} \left\{ \begin{array}{l} \frac{r}{2}\left(t_p^2 - x_{\min}^2\right) \\ +S\left(x_{\max} - t_p\right) - \frac{S}{\beta} + \frac{r}{\beta} t_{o_2} + \frac{r\gamma}{2}\left(x_{\max} - t_{o_2}\right)^2 \end{array} \right\}$$

$$\bar{I}_{r_2} = \frac{\frac{1}{2}}{x_{\max} - x_{\min}} \left\{ \begin{array}{l} -\frac{1}{3r^2}(S-W)^3 \\ +\frac{x_{\max}}{r}(S-W)^2 \\ -(S-W)x_{\min}^2 + \frac{r}{3} x_{\min}^3 \end{array} \right\}$$

$$\bar{I}_{o_2} = \frac{1}{x_{\max} - x_{\min}} \left\{ \begin{array}{l} \frac{1}{6r^2}(S-W)^3 - \frac{x_{\max}}{2r}(S-W)^2 \\ +\frac{S}{2} t_p^2 \\ +\left(\frac{S}{\beta} - S t_p - \frac{r}{2} t_p^2\right)\left(x_{\max} - t_p\right) \end{array} \right.$$

$$\left. \begin{array}{l} +\frac{1}{\beta^2}\left(r t_{o_2} - S\right) - \frac{r}{2\beta} t_{o_2}\left(2x_{\max} - t_{o_2}\right) \\ -\frac{W}{2} x_{\min}^2 + \frac{r}{2\beta} t_p^2 - \frac{r}{6} t_p^3 \end{array} \right\}$$

$$\bar{D}_2 = \frac{1}{x_{\max} - x_{\min}} \left\{ \begin{array}{l} S\left(t_{o_2} - t_p\right) - \frac{S}{\beta} + \frac{r}{\beta} t_{o_2} \\ -\frac{r}{2}\left(t_{o_2}^2 - t_p^2\right) \\ +\left(S - r t_{o_2}\right)\left(x_{\max} - t_{o_2}\right) \end{array} \right\}$$

$$\bar{B}_2 = \frac{1}{x_{\max} - x_{\min}} \frac{\gamma r}{6}\left(x_{\max} - t_{o_2}\right)^3$$

$$\bar{L}_2 = \frac{1}{x_{\max} - x_{\min}} \frac{(1-\gamma) r}{2}\left(x_{\max} - t_{o_2}\right)^2$$

Case (iii) $t_{r_3} \le t_{o_3} \le t_p$

$$\bar{Q}_3 = \frac{\frac{1}{2}}{x_{\max} - x_{\min}} \left\{ r\gamma\left(x_{\max} - \frac{S}{r}\right)^2 - \frac{S^2}{r} + 2S x_{\max} - r x_{\min}^2 \right\}$$

$$\bar{I}_{r_3} = \frac{\frac{1}{2}}{x_{\max} - x_{\min}} \left\{ \begin{array}{l} -\frac{(S-W)^3}{3r^2} + \frac{(S-W)^2}{r} x_{\max} \\ -(S-W)x_{\min}^2 + \frac{r}{3} x_{\min}^3 \end{array} \right\}$$

$$\bar{I}_{o_3} = \frac{\frac{1}{2}}{x_{\max} - x_{\min}} \left\{ \begin{array}{l} -\frac{W\left(3S^2 - 3SW + W^2\right)}{3r^2} \\ +\frac{W(2S-W)}{r} x_{\max} - W x_{\min}^2 \end{array} \right\}$$

$$\bar{B}_3 = \frac{\frac{1}{2}}{x_{\max} - x_{\min}} \frac{\gamma r}{3}\left(x_{\max} - \frac{S}{r}\right)^3$$

$$\bar{L}_3 = \frac{\frac{1}{2}}{x_{\max} - x_{\min}} r(1-\gamma)\left(x_{\max} - \frac{S}{r}\right)^2$$

Appendix 2.

Case (i) $t_p \le t_{r_1} \le t_{o_1}$

$$\bar{Q}_1 = \frac{1}{\Phi(x_{\max})-\Phi(x_{\min})}\left\{ \begin{array}{l} \frac{r\sigma}{\sqrt{2\pi}}\left(e^{-\frac{(-\rho+x_{\min})^2}{2\sigma^2}} - e^{-\frac{\left(-\rho+t_p\right)^2}{2\sigma^2}}\right) \\ +\frac{\gamma r\sigma}{\sqrt{2\pi}}\left(e^{-\frac{\left(-\rho+t_{o_1}\right)^2}{2\sigma^2}} - e^{-\frac{(-\rho+x_{\max})^2}{2\sigma^2}}\right) \end{array}\right.$$

$$-\frac{r\gamma(\rho-S)}{2}Erf\left[\frac{-\rho+t_{o_1}}{\sqrt{2}\sigma}\right] + \frac{S+r\gamma(\rho-S)}{2}Erf\left[\frac{-\rho+x_{\max}}{\sqrt{2}\sigma}\right] - \frac{r\rho}{2}Erf\left[\frac{-\rho+x_{\min}}{\sqrt{2}\sigma}\right]$$

$$+\frac{S\alpha+r-W\alpha-r\alpha t_p}{2\alpha}e^{\frac{\alpha}{2}\left(-2\rho+2t_p+\alpha\sigma^2\right)}\left(Erf\left[\frac{-\rho+t_p+\alpha\sigma^2}{\sqrt{2}\sigma}\right] - Erf\left[\frac{-\rho+t_{r_1}+\alpha\sigma^2}{\sqrt{2}\sigma}\right]\right)$$

$$-\frac{S\alpha+r-r\alpha\rho}{2\alpha}Erf\left[\frac{-\rho+t_p}{\sqrt{2}\sigma}\right] - \frac{r(\alpha-\beta)}{2\alpha\beta}\left(Erf\left[\frac{-\rho+t_{r_1}}{\sqrt{2}\sigma}\right] + Erf\left[\frac{-\rho+t_{o_1}}{\sqrt{2}\sigma}\right]\right)$$

$$+\frac{W}{2}e^{\frac{\beta}{2}\left(-2\rho+2t_p+\beta\sigma^2\right)}\left(Erf\left[\frac{-\rho+t_p+\beta\sigma^2}{\sqrt{2}\sigma}\right] - Erf\left[\frac{-\rho+t_{o_1}+\beta\sigma^2}{\sqrt{2}\sigma}\right]\right)$$

$$+\frac{r}{2\beta}e^{\frac{\beta}{2}\left(-2\rho+2t_p+\beta\sigma^2\right)}\left(Erf\left[\frac{-\rho+t_{r_1}+\beta\sigma^2}{\sqrt{2}\sigma}\right] - Erf\left[\frac{-\rho+t_{o_1}+\beta\sigma^2}{\sqrt{2}\sigma}\right]\right)\left(\frac{r+\alpha\left(S-W-rt_p\right)}{r}\right)^{\frac{\beta}{\alpha}}$$

$$+\left(\begin{array}{l}\frac{(S-W)\left(\alpha\rho-2\alpha t_p-2\right)}{4\alpha} \\ +\frac{r\left(2(\alpha\rho-1)+\alpha^2(t_p^2-\rho^2-\sigma^2)\right)}{4\alpha^2}\end{array}\right)Erf\left[\frac{-\rho+t_p}{\sqrt{2}\sigma}\right]\Bigg\}$$

$$\bar{I}_{o_1} = \frac{1}{\Phi\left(x_{\max}\right)-\Phi\left(x_{\min}\right)}\left\{\begin{array}{l} \frac{r\sigma}{\sqrt{2\pi}\beta}\left(e^{-\frac{\left(-\rho+t_{o_1}\right)^2}{2\sigma^2}} - e^{-\frac{\left(-\rho+t_{r_1}\right)^2}{2\sigma^2}}\right) \\ +\frac{W\sigma}{\sqrt{2\pi}}\left(e^{-\frac{(-\rho+x_{\min})^2}{2\sigma^2}} - e^{-\frac{\left(-\rho+t_p\right)^2}{2\sigma^2}}\right)\end{array}\right.$$

$$+\frac{W}{2\beta}\left(\beta(\rho-t_p)-1\right)Erf\left[\frac{-\rho+t_p}{\sqrt{2}\sigma}\right] - \frac{W\rho}{2}Erf\left[\frac{-\rho+x_{\min}}{\sqrt{2}\sigma}\right] + \frac{r\left(1+\beta(t_{o_1}-\rho)\right)}{2\beta^2}Erf\left[\frac{-\rho+t_{o_1}}{\sqrt{2}\sigma}\right]$$

$$+\frac{W\left(1+\beta t_p\right)-r\left(t_{o_1}-t_{r_1}\right)}{2\beta}Erf\left[\frac{-\rho+x_{\max}}{\sqrt{2}\sigma}\right] - \frac{r\left(1+\beta(t_{r_1}-\rho)\right)}{2\beta^2}Erf\left[\frac{-\rho+t_{r_1}}{\sqrt{2}\sigma}\right]$$

$$+\frac{r}{2\beta^2}e^{\frac{\beta}{2}\left(-2\rho+2t_p+\beta\sigma^2\right)}\left(Erf\left[\frac{-\rho+t_{r_1}+\beta\sigma^2}{\sqrt{2}\sigma}\right] - Erf\left[\frac{-\rho+t_{o_1}+\beta\sigma^2}{\sqrt{2}\sigma}\right]\right)\left(\frac{r+\alpha\left(S-W-rt_p\right)}{r}\right)^{\frac{\beta}{\alpha}}$$

$$+\frac{W}{2\beta}e^{\frac{\beta}{2}\left(-2\rho+2t_p+\beta\sigma^2\right)}\left(\begin{array}{l}Erf\left[\frac{-\rho+t_p+\beta\sigma^2}{\sqrt{2}\sigma}\right] \\ -Erf\left[\frac{-\rho+t_{o_1}+\beta\sigma^2}{\sqrt{2}\sigma}\right]\end{array}\right)\Bigg\}$$

$$\bar{I}_{r_1} = \frac{1}{\Phi\left(x_{\max}\right)-\Phi\left(x_{\min}\right)}\left\{\begin{array}{l}\frac{r\sigma}{\sqrt{2\pi}\alpha} \\ \left(\begin{array}{l} e^{-\frac{\left(-\rho+t_{r_1}\right)^2}{2\sigma^2}} \\ -\frac{2-\alpha t_p}{2}e^{-\frac{\left(-\rho+t_p\right)^2}{2\sigma^2}} \\ -\frac{\alpha}{2}e^{-\frac{(-\rho+x_{\min})^2}{2\sigma^2}}\end{array}\right)\end{array}\right.$$

$$+\frac{\sigma}{2\sqrt{2\pi}}(-2S+2W+r\rho)\left(e^{-\frac{\left(-\rho+t_p\right)^2}{2\sigma^2}} - e^{-\frac{(-\rho+x_{\min})^2}{2\sigma^2}}\right) + \frac{r}{2\alpha^2}\left(1+\alpha\left(t_{r_1}+t_p-\rho\right)\right)Erf\left[\frac{-\rho+t_{r_1}}{\sqrt{2}\sigma}\right]$$

$$+\frac{S-W+\frac{r}{\alpha}-rt_p}{2\alpha}e^{\frac{\alpha}{2}\left(-2\rho+2t_p+\alpha\sigma^2\right)}\left(Erf\left[\frac{-\rho+t_p+\alpha\sigma^2}{\sqrt{2}\sigma}\right] - Erf\left[\frac{-\rho+t_{r_1}+\alpha\sigma^2}{\sqrt{2}\sigma}\right]\right)$$

$$-\frac{\left(\rho(S-W)-\frac{r}{2}\left(\rho^2+\sigma^2\right)\right)}{2}Erf\left[\frac{-\rho+x_{\min}}{\sqrt{2}\sigma}\right] + \frac{(S-W)\left(1+\alpha t_p\right)-r\left(t_{r_1}-t_p-\frac{\alpha t_p^2}{2}\right)}{\alpha}Erf\left[\frac{-\rho+x_{\max}}{\sqrt{2}\sigma}\right]$$

$$\bar{D}_1 = \frac{1}{\Phi\left(x_{\max}\right)-\Phi\left(x_{\min}\right)}\left\{\begin{array}{l}\frac{r\sigma}{\sqrt{2\pi}}\left(\begin{array}{l}e^{-\frac{\left(-\rho+t_{o_1}\right)^2}{2\sigma^2}} \\ -e^{-\frac{\left(-\rho+t_p\right)^2}{2\sigma^2}}\end{array}\right) \\ -\frac{S\alpha+r-r\alpha\rho}{2\alpha}Erf\left[\frac{-\rho+t_p}{\sqrt{2}\sigma}\right]\end{array}\right.$$

$$+\frac{S-W+\frac{r}{\alpha}-rt_p}{2}e^{\frac{\alpha}{2}\left(-2\rho+2t_p+\alpha\sigma^2\right)}\left(Erf\left[\frac{-\rho+t_p+\alpha\sigma^2}{\sqrt{2}\sigma}\right] - Erf\left[\frac{-\rho+t_{r_1}+\alpha\sigma^2}{\sqrt{2}\sigma}\right]\right)$$

$$+\frac{r(1-\beta\rho)}{2\beta}Erf\left[\frac{-\rho+t_{o_1}}{\sqrt{2}\sigma}\right] + \frac{S-rt_{o_1}}{2}Erf\left[\frac{-\rho+x_{\max}}{\sqrt{2}\sigma}\right] - \frac{r(\alpha-\beta)}{2\alpha\beta}Erf\left[\frac{-\rho+t_{r_1}}{\sqrt{2}\sigma}\right]$$

$$+\frac{r}{2\beta}e^{\frac{\beta}{2}\left(-2\rho+2t_p+\beta\sigma^2\right)}\left(\begin{array}{l}Erf\left[\frac{-\rho+t_{r_1}+\beta\sigma^2}{\sqrt{2}\sigma}\right] \\ -Erf\left[\frac{-\rho+t_{o_1}+\beta\sigma^2}{\sqrt{2}\sigma}\right]\end{array}\right)\left(\frac{r+\alpha\left(S-W-rt_p\right)}{r}\right)^{\frac{\beta}{\alpha}}$$

$$+\frac{W}{2}e^{\frac{\beta}{2}\left(-2\rho+2t_p+\beta\sigma^2\right)}\left(\begin{array}{l} Erf\left[\frac{-\rho+t_p+\beta\sigma^2}{\sqrt{2}\sigma}\right] \\ -Erf\left[\frac{-\rho+t_{o_1}+\beta\sigma^2}{\sqrt{2}\sigma}\right]\end{array}\right)\Bigg\}$$

$$\bar{B}_1 = \frac{\frac{r\gamma}{2\sqrt{2\pi}}}{\Phi\left(x_{\max}\right)-\Phi\left(x_{\min}\right)}\left\{\begin{array}{l}\sigma\left(\rho-t_{o_1}\right) \\ e^{-\frac{\left(-\rho+t_{o_1}\right)^2}{2\sigma^2}} \\ -\sigma\left(\rho+x_{\max}-2t_{o_1}\right) \\ e^{-\frac{\left(-\rho+x_{\max}\right)^2}{2\sigma^2}}\end{array}\right.$$

$$+\sqrt{\frac{\pi}{2}}\left(Erf\left[\frac{-\rho+x_{\max}}{\sqrt{2}\sigma}\right]-Erf\left[\frac{-\rho+t_{o_1}}{\sqrt{2}\sigma}\right]\right)\left(\left(t_{o_1}-\rho\right)^2+\sigma^2\right)\Bigg\}$$

$$\bar{L}_1 = \frac{\frac{(1-\gamma)r}{\sqrt{2\pi}}}{\Phi\left(x_{\max}\right)-\Phi\left(x_{\min}\right)}\left\{\left(t_{o_2}-\rho\right)\sqrt{\frac{\pi}{2}}\left(\begin{array}{l} Erf\left[\frac{-\rho+t_{o_1}}{\sqrt{2}\sigma}\right] \\ -Erf\left[\frac{-\rho+x_{\max}}{\sqrt{2}\sigma}\right]\end{array}\right)\right.$$

$$\left.+e^{-\frac{\left(-\rho+t_{o_1}\right)^2}{2\sigma^2}}-e^{-\frac{\left(\rho-x_{\max}\right)^2}{2\sigma^2}}\right\}$$

$$Erf(t)=\frac{2}{\sqrt{\pi}}\int_0^t e^{-x^2}\,dx.$$

Case (ii) $t_{r_2}\le t_p\le t_{o_2}$

$$\bar{Q}_2 = \frac{1}{\Phi(x_{\max})-\Phi\left(x_{\min}\right)}\left\{\begin{array}{l}\frac{r\sigma}{\sqrt{2\pi}}\left(e^{-\frac{\left(-\rho+x_{\min}\right)^2}{2\sigma^2}}-e^{-\frac{\left(-\rho+t_p\right)^2}{2\sigma^2}}\right) \\ +\frac{\gamma r\sigma}{\sqrt{2\pi}}\left(e^{-\frac{\left(-\rho+t_{o_2}\right)^2}{2\sigma^2}}-e^{-\frac{\left(-\rho+x_{\max}\right)^2}{2\sigma^2}}\right)\end{array}\right.$$

$$\begin{array}{l}+\frac{r\left(1-\beta\gamma\left(\rho-t_{o_2}\right)\right)}{2\beta}Erf\left[\frac{-\rho+t_{o_2}}{\sqrt{2}\sigma}\right] \\ +\frac{S+r\gamma\left(\rho-t_{o_2}\right)}{2}Erf\left[\frac{-\rho+x_{\max}}{\sqrt{2}\sigma}\right] \\ -\frac{r\rho}{2}Erf\left[\frac{-\rho+x_{\min}}{\sqrt{2}\sigma}\right]\end{array}$$

$$\begin{array}{l}+\frac{S\beta+r-r\beta t_p}{2\beta}e^{\frac{\beta}{2}\left(-2\rho+2t_p+\beta\sigma^2\right)} \\ \left(Erf\left[\frac{-\rho+t_p+\beta\sigma^2}{\sqrt{2}\sigma}\right]-Erf\left[\frac{-\rho+t_{o_2}+\beta\sigma^2}{\sqrt{2}\sigma}\right]\right)\end{array}$$

$$-\frac{S\beta+r-r\beta\rho}{2\beta}Erf\left[\frac{-\rho+t_p}{\sqrt{2}\sigma}\right]\Bigg\}$$

$$\bar{I}_{r_2} = \frac{1}{\Phi\left(x_{\max}\right)-\Phi\left(x_{\min}\right)}\left\{-\frac{\sigma}{2\sqrt{2\pi}}\left(\begin{array}{l}(S-W-r\rho) \\ e^{-\frac{(-S+W+r\rho)^2}{2r^2\sigma^2}} \\ -\left(2S-2\rho-r\rho-rx_{\min}\right) \\ e^{-\frac{\left(-\rho+x_{\min}\right)^2}{2\sigma^2}}\end{array}\right)\right.$$

$$\begin{array}{l}+\frac{1}{4}\left(-2\rho(S-W)+r(\rho^2+\sigma^2)\right) \\ \left(Erf\left[\frac{-\rho+x_{\min}}{\sqrt{2}\sigma}\right]+Erf\left[\frac{-S+W+r\rho}{\sqrt{2}r\sigma}\right]\right)\end{array}$$

$$+\frac{(S-W)^2}{4r}\left(\begin{array}{l}Erf\left[\frac{-\rho+x_{\max}}{\sqrt{2}\sigma}\right] \\ +Erf\left[\frac{-S+W+r\rho}{\sqrt{2}r\sigma}\right]\end{array}\right)\Bigg\}$$

$$\bar{I}_{o_2} = \frac{1}{\Phi(x_{\max})-\Phi(x_{\min})}\left\{\frac{\sigma}{2\sqrt{2\pi}}\left(\begin{array}{l}(S-W-r\rho) \\ e^{-\frac{(-S+W+r\rho)^2}{2r^2\sigma^2}} \\ +2We^{-\frac{\left(-\rho+x_{\min}\right)^2}{2\sigma^2}} \\ +2re^{-\frac{\left(-\rho+t_{o_2}\right)^2}{2\sigma^2}}\end{array}\right)\right.$$

$$\begin{array}{l}+\frac{\sigma\left(-2S+r(\rho+t_p-\frac{2}{\beta})\right)}{2\sqrt{2\pi}}e^{-\frac{\left(-\rho+t_p\right)^2}{2\sigma^2}} \\ +\frac{r(t_{o_2}-\rho+\frac{1}{\beta})}{2\beta}Erf\left[\frac{-\rho+t_{o_2}}{\sqrt{2}\sigma}\right] \\ -\frac{W\rho}{2}Erf\left[\frac{-\rho+x_{\min}}{\sqrt{2}\sigma}\right]\end{array}$$

$$+\left(\begin{array}{l}-\frac{(S-W)^2}{2r}+\frac{S-rk}{\beta} \\ +\frac{t_p}{2}(2S-rt_p)\end{array}\right)\frac{1}{2}Erf\left[\frac{-\rho+x_{\max}}{\sqrt{2}\sigma}\right]$$

$$-\left(\begin{array}{l}\frac{2S}{\beta}-\frac{2(\beta\rho-1)}{\beta^2}+2St_p \\ -2S\rho-rt_p^2+r^2(\rho^2+\sigma^2)\end{array}\right)\frac{1}{4}Erf\left[\frac{-\rho+t_p}{\sqrt{2}\sigma}\right]$$

$$+\frac{S\beta+r-r\beta t_p}{2\beta}e^{\frac{\beta}{2}(-2\rho+2t_p+\beta\sigma^2)}\left(\begin{array}{l}Erf\left[\frac{-\rho+t_p+\beta\sigma^2}{\sqrt{2}\sigma}\right] \\ -Erf\left[\frac{-\rho+t_{o_2}+\beta\sigma^2}{\sqrt{2}\sigma}\right]\end{array}\right)$$

$$-\frac{(S-W-r\rho)^2+r^2\sigma^2}{4r}Erf\left[\frac{-S+W+r\rho}{\sqrt{2}r\sigma}\right]\Bigg\}$$

$$\bar{D}_2=\frac{1}{\Phi(x_{\max})-\Phi(x_{\min})}\left\{\begin{array}{l}\frac{r\sigma}{\sqrt{2\pi}}\left(e^{-\frac{(-\rho+t_{o_2})^2}{2\sigma^2}}-e^{-\frac{(-\rho+t_p)^2}{2\sigma^2}}\right)\\-\frac{S\beta+r-r\beta\rho}{2\beta}Erf\left[\frac{-\rho+t_p}{\sqrt{2}\sigma}\right]\end{array}\right.$$

$$+\frac{S\beta+r-r\beta t_p}{2\beta}e^{\frac{\beta}{2}(-2\rho+2t_p+\beta\sigma^2)}\left(\begin{array}{l}Erf\left[\frac{-\rho+t_p+\beta\sigma^2}{\sqrt{2}\sigma}\right]\\-Erf\left[\frac{-\rho+t_{o_2}+\beta\sigma^2}{\sqrt{2}\sigma}\right]\end{array}\right)$$

$$+\frac{r(1-\beta\rho+\beta t_{o_2})}{2\beta}Erf\left[\frac{-\rho+t_{o_2}}{\sqrt{2}\sigma}\right]+\frac{S-rt_{o_2}}{2}Erf\left[\frac{-\rho+x_{\max}}{\sqrt{2}\sigma}\right]\Bigg\}$$

$$\bar{B}_2=\frac{\frac{r\gamma}{2\sqrt{2\pi}}}{\Phi(x_{\max})-\Phi(x_{\min})}\left\{\begin{array}{l}\sigma(\rho-t_{o_2})e^{-\frac{(-\rho+t_{o_2})^2}{2\sigma^2}}\\-\sigma(\rho+x_{\max}-2t_{o_2})e^{-\frac{(-\rho+x_{\max})^2}{2\sigma^2}}\end{array}\right.$$

$$+\sqrt{\frac{\pi}{2}}\left(Erf\left[\frac{-\rho+x_{\max}}{\sqrt{2}\sigma}\right]-Erf\left[\frac{-\rho+t_o}{\sqrt{2}\sigma}\right]\right)((t_{o_2}-\rho)^2+\sigma^2)\Bigg\}$$

$$\bar{L}_2=\frac{\frac{(1-\gamma)r}{\sqrt{2\pi}}}{\Phi(x_{\max})-\Phi(x_{\min})}\left\{(t_{o_2}-\rho)\sqrt{\frac{\pi}{2}}\left(\begin{array}{l}Erf\left[\frac{-\rho+t_{o_2}}{\sqrt{2}\sigma}\right]\\-Erf\left[\frac{-\rho+x_{\max}}{\sqrt{2}\sigma}\right]\end{array}\right)\right.$$

$$+e^{-\frac{(-\rho+t_{o_2})^2}{2\sigma^2}}-e^{-\frac{(\rho-x_{\max})^2}{2\sigma^2}}\Bigg\}$$

Case (iii) $t_{r_3}\le t_{o_3}\le t_p$

$$\bar{Q}_3=\frac{1}{\Phi(x_{\max})-\Phi(x_{\min})}\left\{\begin{array}{l}-r\rho\frac{1}{2}Erf\left[\frac{-\rho+x_{\min}}{\sqrt{2}\sigma}\right]\\+(S-S\gamma+r\rho\gamma)\\\frac{1}{2}Erf\left[\frac{-\rho+x_{\max}}{\sqrt{2}\sigma}\right]\end{array}\right.$$

$$\left.\begin{array}{l}+(\gamma-1)(S-r\rho)\frac{1}{2}Erf\left[\frac{S-r\rho}{\sqrt{2}r\sigma}\right]\\+\frac{r\sigma}{\sqrt{2\pi}}\left(\begin{array}{l}(\gamma-1)e^{-\frac{(S-r\rho)^2}{2r^2\sigma^2}}\\-\gamma e^{-\frac{(\rho-x_{\max})^2}{2\sigma^2}}+e^{-\frac{(\rho-x_{\min})^2}{2\sigma^2}}\end{array}\right)\end{array}\right\}$$

$$\bar{I}_{r_3}=\frac{1}{\Phi(x_{\max})-\Phi(x_{\min})}\left\{\begin{array}{l}((-S+W+r\rho)^2+r^2\sigma^2)\\\frac{1}{4r}Erf\left[\frac{-S+W+r\rho}{\sqrt{2}r\sigma}\right]\\+\frac{(S-W)^2}{4r}\\Erf\left[\frac{-\rho+x_{\max}}{\sqrt{2}\sigma}\right]\end{array}\right.$$

$$\begin{array}{l}+\frac{1}{4}(2\rho(-S+W)+r(\rho^2+\sigma^2))\\Erf\left[\frac{-\rho+x_{\min}}{\sqrt{2}\sigma}\right]+\frac{\sigma}{2\sqrt{2\pi}}(-S+W+r\rho)\\e^{-\frac{(-S+W+r\rho)^2}{2r^2\sigma^2}}\end{array}$$

$$+\frac{\sigma}{2\sqrt{2\pi}}(2(S-W)-r(\rho+x_{\min}))e^{-\frac{(\rho-x_{\min})^2}{2\sigma^2}}\Bigg\}$$

$$\bar{I}_{o_3}=\frac{1}{\Phi(x_{\max})-\Phi(x_{\min})}\left\{\begin{array}{l}-((S-W)^2\\-2r\rho(rS-W)\\+r^2(\rho^2+\sigma^2))\\\frac{1}{4r}Erf\left[\frac{-S+W+r\rho}{\sqrt{2}r\sigma}\right]\end{array}\right.$$

$$\begin{array}{l}-((S-r\rho)^2+r^2\sigma^2)\frac{1}{4r}Erf\left[\frac{S-r\rho}{\sqrt{2}r\sigma}\right]\\+W(2S-W)\frac{1}{4r}Erf\left[\frac{-\rho+x_{\max}}{\sqrt{2}\sigma}\right]\\-W\rho\frac{1}{2}Erf\left[\frac{-\rho+x_{\min}}{\sqrt{2}\sigma}\right]\end{array}$$

$$\left.\begin{array}{l}-\frac{\sigma}{2\sqrt{2\pi}}(-S+W+r\rho)e^{-\frac{(-S+W+r\rho)^2}{2r^2\sigma^2}}\\+\frac{\sigma}{2\sqrt{2\pi}}(-S+r\rho)e^{-\frac{(-s+r\rho)^2}{2r^2\sigma^2}}+\frac{W\sigma}{\sqrt{2\pi}}e^{-\frac{(\rho-x_{\min})^2}{2\sigma^2}}\end{array}\right\}$$

$$\bar{B}_3=\frac{\gamma}{\Phi(x_{\max})-\Phi(x_{\min})}\left\{\begin{array}{l}\frac{1}{4r}((S-r\rho)^2+r^2\sigma^2)\\\left(Erf\left[\frac{x_{\max}-\rho}{\sqrt{2}\sigma}\right]-Erf\left[\frac{S-r\rho}{\sqrt{2}r\sigma}\right]\right)\end{array}\right.$$

$$+\frac{\sigma}{2\sqrt{2\pi}}(S-rx_{\max})e^{-\frac{(S-r\rho)^2}{2r^2\sigma^2}}\Bigg\}$$

$$\bar{L}_3=\frac{(1-\gamma)}{\Phi(x_{\max})-\Phi(x_{\min})}\left\{\begin{array}{l}(S-r\rho)\\\left(Erf[\frac{x_{\max}-\rho}{\sqrt{2}\sigma}]-Erf[\frac{S-r\rho}{\sqrt{2}r\sigma}]\right)\\+\frac{r\sigma}{\sqrt{2\pi}}\left(e^{-\frac{(\rho-x_{\max})^2}{2\sigma^2}}-e^{-\frac{(S-r\rho)^2}{2r^2\sigma^2}}\right)\end{array}\right\}$$