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BAYESIAN ESTIMATION UNDER POISSON TESTING USING THE LINEX LOSS FUNCTION

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Abstract- **Bayesian estimation for the mean failure rate under Poisson type 1 testing is considered. The loss function that is considered in this study is the LINEX loss. The admissibility of the linear function of the sample mean (failure rate) for the Poisson testing is also discussed.**

INTRODUCTION

Consider n items which are placed on test that is truncated after a specified time T. Suppose the failure time T_i for item i is exponential with parameter λ, where λ is the failure rate and $T = \sum_{i=1}^{n} T_i$. Then the total number of failures r for the total cummulative time T has a Poisson distribution with parameter λT [1]. The parameter of interest in the Poisson testing is λ where λ is assumed to have a gamma probability density function with parameters δ and ρ. The loss function that will be considered is that proposed by Varian [2] called the LINEX loss function. Varian used the LINEX loss to assess real estate, where litigation costs due to overestimation is much larger than the revenue loss due to underestimation. Zellner [3] gave an example relating to the construction of dams in which case underestimation of flood water level is usually much more serious than overestimation. The functional form of the LINEX loss function is of the form:

$L(\Delta) = b\exp(a\Delta) - c\Delta - b$ for $a, c \neq 0$, $b > 0$ and $\Delta = \lambda - \hat{\lambda}$ is the error in estimating λ. $L(0) = 0$ so the minimum is attained at $\Delta = 0$ and hence $ab = c$. Thus $L(\Delta)$ can be written as

$$L(\Delta) = b[\exp(a\Delta) - a\Delta - 1], \quad a \neq 0 \text{ and } b > 0 \tag{1}$$

Following are some properties of the LINEX loss as given in equation (1):

(1) $L(\Delta)$ is an asymmetric or convex loss function

(2) b serves to scale the loss function

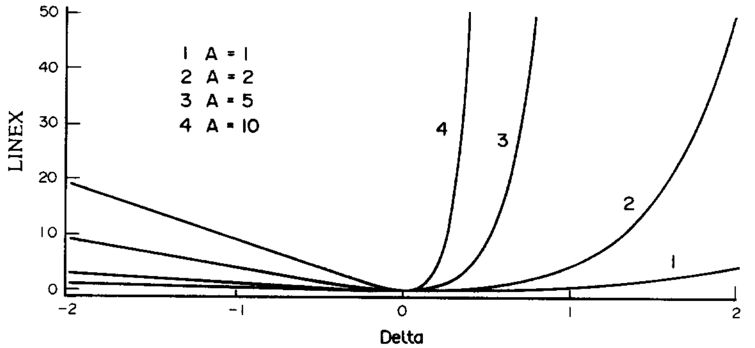

Fig. 1. LINEX loss function.

(3) a determines the shape of the loss function

(4) for $a > 0$, the loss function increases almost linearly for $\Delta < 0$

(5) for $a > 0$, the loss function increases almost exponentially for $\Delta > 0$

(6) properties (4) and (5) imply that for $a > 0$, under-estimation is a more serious mistake than over-estimation

(7) for $a < 0$, the loss function increases almost exponentially for $\Delta < 0$

(8) for $a < 0$, the loss function increases almost linearly for $\Delta > 0$

(9) properties (7) and (8) imply that for $a < 0$, over-estimation is a more serious mistake than under-estimation

(10) for small values of a, the loss is approximately the squared error loss $\frac{a^2}{2}(\lambda - \hat{\lambda})^2$

Figure 1 illustrates these properties.

ASSUMPTIONS

1. The failure rate λ has a gamma distribution with shape parameter δ and scale parameter ρ. That is, the prior distribution for λ is of the form

$$h(\lambda) = \frac{\rho^{\delta} \lambda^{\delta-1} \exp[-\rho\lambda]}{\Gamma(\delta)}, \quad \delta \geq 0 \text{ and } \rho \geq 0 \tag{2}$$

2. The prior distribution for the number of failures r, given the failure rate λ, is Poisson with parameter λT. That is,

$$f(r \mid \lambda) = \frac{(\lambda T)^r \exp[-\lambda T]}{r!}, \quad r = 0, 1, 2, \ldots . \tag{3}$$

3. The posterior distribution for λ given r is gamma with scale parameter $\rho + T$ and shape parameter $\delta + r$.

BAYES ESTIMATE OF λ UNDER THE LINEX LOSS

The risk is the posterior expected loss given by

$$R(\lambda,\hat{\lambda}) = E^{\lambda|r}[\, L(\Delta)\,] \tag{4}$$

The value of $\hat{\lambda}$, $\hat{\lambda}_{BL}$, that minimizes (4) is the Bayes estimate of λ relative to the LINEX loss function. Now,

$$\frac{d}{d\hat{\lambda}} R(\lambda,\hat{\lambda}) = 0$$

implies

$$E^{\lambda|r}[\exp[a(\lambda - \hat{\lambda}_{BL})]] = 1$$

from which,

$$\hat{\lambda}_{BL} = - a^{-1} \ln [\, \frac{(\rho + T - a)^{\delta+r}}{(\rho + T)}\,] \tag{5}$$

BAYES ESTIMATE OF λ UNDER SQUARE ERROR LOSS

The squared error loss is given by

$$L(\Delta) = \Delta^2$$

from which the Bayes estimate, $\hat{\lambda}_{BS}$, relative to the squared error loss is the solution of

$$\hat{\lambda}_{BS} = E^{\lambda|r}(\lambda)$$

From which

$$\hat{\lambda}_{BS} = \frac{(r + \delta)}{(\rho + T)} \tag{6}$$

RISKS OF $\hat{\lambda}_{BS}$ AND $\hat{\lambda}_{BL}$ RELATIVE TO THE LINEX LOSS FUNCTION

Now, the risks relative to the LINEX loss function are

$$R_L(\hat{\lambda}_{BS}) = E^{\lambda|r}[\, \exp\{\, a(\lambda - \hat{\lambda}_{BS})\} - a\,(\lambda - \hat{\lambda}_{BS}) - 1\,]$$

$$= \exp[\, - a \frac{(r + \delta)}{(\rho + T)}]\, [\, \frac{(\rho + T)}{(\rho + T\text{-}a)} - 1\,] \tag{7}$$

and

$$R_L(\hat{\lambda}_{BL}) = a \frac{(r + \delta)}{(\rho + T)} - \ln [\, \frac{(\rho + T - a)^{\delta+r}}{(\rho + T)}] \tag{8}$$

Now, the risk difference scaled by b = 1 is

$$R_L(\hat{\lambda}_{BL}) - R_L(\hat{\lambda}_{BS}) = - a \frac{(r + \delta)}{(\rho + T)} - \ln [\, \frac{(\rho + T - a)^{\delta+r}}{(\rho + T)}]$$

$$- \exp[-a \frac{(r+\delta)}{(\rho+T)}] [\frac{(\rho+T)}{(\rho+T-a)^{\delta+r}}] + 1$$

$$< 0 \text{ for } a > 0 \text{ and } \rho + T > a + 1 \tag{9}$$

That is, $R_L(\hat{\lambda}_{BL})$ is uniformly smaller than $R_L(\hat{\lambda}_{BS})$. Thus, $\hat{\lambda}_{BL}$ uniformly dominates $\hat{\lambda}_{BS}$ for $a > 0$ and thus $\hat{\lambda}_{BS}$ is inadmissible relative to the LINEX loss function when $a > 0$. The difference in the risk can be substantial. Thus employing the appropriate loss function is important. Figure 2 gives a graphical representation of equation (9) for specific parameters.

RISKS OF $\hat{\lambda}_{BS}$ AND $\hat{\lambda}_{BL}$ RELATIVE TO THE SQUARE ERROR LOSS FUNCTION

The risks relative to the squared error loss are

$$R_S(\lambda, \hat{\lambda}_{BS}) = \frac{(\delta + r)}{(\rho + T)^2} \tag{10}$$

and

$$R_S(\lambda, \hat{\lambda}_{BL}) = \frac{(\delta + r + 1)(\delta + r)}{(\rho + T)^2} + 2a^{-1} \ln [\frac{(\rho + T - a)^{\delta+r}}{(\rho + T)}] (\frac{\delta + r}{\rho + T})$$

$$+ a^{-2} \ln^2 [\frac{(\rho + T - a)^{\delta+r}}{\rho + T}] \tag{11}$$

Now, the risk difference scaled by $b = 1$ is

$$R_S(\lambda, \hat{\lambda}_{BL}) - R_S(\lambda, \hat{\lambda}) = \frac{(\delta + r)^2}{(\rho + T)^2} + 2a^{-1} \frac{(\delta + r)}{(\rho + T)} \ln[\frac{(\rho + T - a)^{\delta+r}}{(\rho + T)}]$$

$$+ a^{-2} \ln^2 [\frac{(\rho + T - a)^{\delta+r}}{(\rho + T)}]$$

$$> 0 \text{ for } a > 0 \text{ and } \rho+T \geq 1+a \tag{12}$$

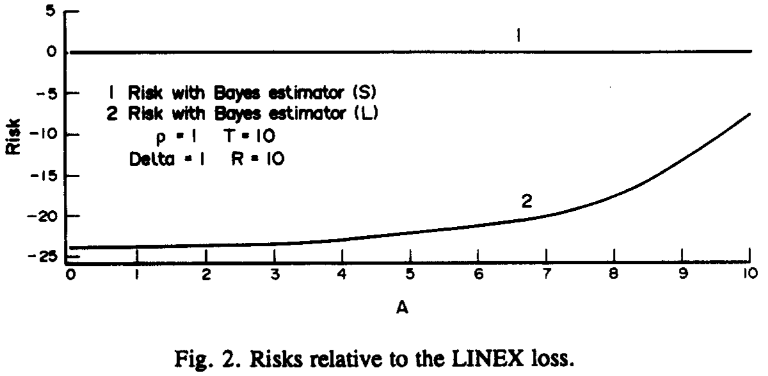

Fig. 2. Risks relative to the LINEX loss.

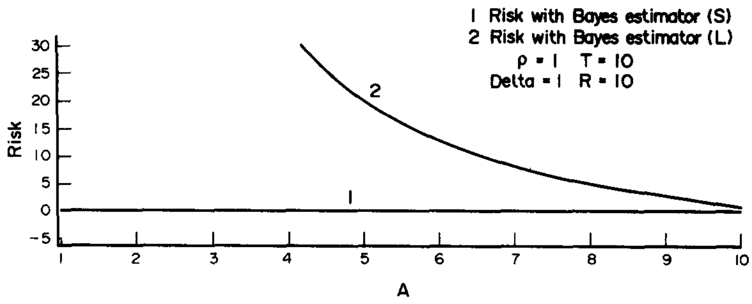

Fig. 3. Risks relative to the squared error loss.

Thus $R_S(\lambda, \hat{\lambda}_{BS})$ is uniformly smaller than $R_S(\lambda, \hat{\lambda}_{BL})$ for $a > 0$ and $\rho + T \geq 1 + a$. Therefore $\hat{\lambda}_{BS}$ uniformly dominates $\hat{\lambda}_{BL}$ when $a > 0$ and $\rho + T \geq 1 + a$ and thus $\hat{\lambda}_{BL}$ is inadmissible relative to the squared error loss. Figure 3 gives a graphical representation of equation (12) for specific parameters.

ADMISSIBILITY OF THE AVERAGE FAILURES

The likelihood function

$$L(\underset{\sim}{r} \mid \lambda) \ \alpha \ (\lambda)^{n\bar{r}} \tag{13}$$

where $\bar{r}$ is the average failure during time T. Equation (13) is obtained from the joint marginals. Since λ has a gamma distribution with shape parameter δ and scale parameter ρ, then the posterior distribution for λ given $\underset{\sim}{r}$ (the vector of failures) is

$$f(\lambda \mid \underset{\sim}{r}) \ \alpha \ (\lambda)^{n\bar{r}} \lambda^{\delta-1} \exp[-\rho\lambda]$$

$$\text{or} \quad f(\lambda \mid \underset{\sim}{r}) = \frac{\rho^{n\bar{r}+\delta}}{\Gamma(n\bar{r}+\delta)} \lambda^{n\bar{r}+\delta-1} \exp[-\rho\lambda] \tag{14}$$

That is, the posterior distribution of λ given $\underset{\sim}{r}$ is gamma with shape parameter $n\bar{r}+\delta$ and scale parameter ρ.

Now

$$\hat{\lambda}_{BL} = a^{-1} \ln [E^{\lambda \mid \underset{\sim}{r}} [\exp(a\lambda)]]$$

$$= a^{-1} [n\bar{r} \ln [\frac{1}{1-\frac{a}{\rho}}] + \delta \ln [\frac{1}{1-\frac{a}{\rho}}]] \tag{15}$$

Let $\hat{\lambda} = \alpha\bar{r} + \beta$ and consider $R(\lambda, \alpha\bar{r} + \beta) = E[L(\Delta)]$ with respect to the Poisson distribution with mean λT. Rojo [4] and Alvandi [5] considered the sample of failures (r_i, i =1, ..., n) to be coming from a normal population. Now,

$$R(\lambda, \alpha\bar{r}+\beta) = \exp[a(\beta - \lambda)]\, E[\exp(a\alpha\bar{r})] - a(\beta - \lambda) - a\alpha E(\bar{r}) - 1 \tag{16}$$

Let $t = a\alpha$, then

$$\begin{aligned} E[\exp(a\alpha\bar{r})] &= E[\exp(t\bar{r})] \\ &= E[\exp(\tfrac{t}{n}\sum_{i=1}^{n} r_i)] \\ &= \prod_{i=1}^{n} E[\exp(\tfrac{t}{n} r_i)] \\ &= \prod_{i=1}^{n} M_{r_i}(\tfrac{t}{n}) \end{aligned}$$

Now since the posterior distribution of r_i given λ is Poisson with parameter λT, then

$$M_{r_i}(t) = \exp[\lambda T(\exp(t) - 1)]$$

Thus

$$E[\exp(a\alpha\bar{r})] = \prod_{i=1}^{n} \exp[\lambda T(\exp(\tfrac{t}{n}) - 1)]$$

$$= \exp[n\lambda T(\exp(\tfrac{a\alpha}{n} - 1))] \tag{17}$$

Also, $E(\bar{r}) = \lambda T$, therefore

$$\begin{aligned} R(\lambda, \alpha\bar{r} + \beta) = {} & \exp[a(\beta - \lambda)]\exp[n\lambda T(\exp(\tfrac{a\alpha}{n} - 1))] \\ & - a(\beta - \lambda) - a\alpha\lambda T - 1 \end{aligned} \tag{18}$$

Now $\frac{dR}{d\alpha}[R(\lambda, \alpha\bar{r} + \beta)] = a\lambda T[\exp[[a(\beta - \lambda)] + \frac{a\alpha}{n} + n\lambda T[\exp(\frac{a\alpha}{n} - 1)]] - 1]$.

Let $g(\lambda) = a(\beta - \lambda) + \frac{a\alpha}{n} + n\lambda T[\exp(\frac{a\alpha}{n} - 1)] - 1$, for $\lambda > 0$, then

$\frac{d}{d\lambda}[g(\lambda)] = -a + nT[\exp(\frac{a\alpha}{n}) - 1] = 0$,

which implies that $\alpha = a^{-1} n \ln[\frac{a}{nT} + 1]$.

Thus $g(\lambda)$ is an increasing function whenever $\alpha > a^{-1} n \ln[\frac{a}{nT} + 1]$. This implies that

$R(\lambda, \alpha\bar{r} + \beta)$ is an increasing function of α whenever $\alpha > a^{-1} n \ln[\frac{a}{nT} + 1]$. Moreover,

$R(\lambda, \alpha\bar{r} + \beta)$ is a continuous function.

From the preceeding discussion we propose the following theorem:

Theorem: Let $T < \frac{\rho - a}{n}$ and let λ be distributed as a gamma distribution with shape parameter δ and scale parameter ρ, then $\alpha\bar{r} + \beta$ is admissible whenever $0 < \alpha < a^{-1} n \ln[\frac{a}{nT} + 1]$.

Proof: Recall $\hat{\lambda}_{BL} = a^{-1}\,[n\bar{r}\,\ln[\frac{1}{1-\frac{a}{\rho}}] + \delta\ln[\frac{1}{1-\frac{a}{\rho}}]]$

$$= [a^{-1}\,n\ln[\frac{1}{1-\frac{a}{\rho}}]]\,\bar{r} + [a^{-1}\,\delta\ln[\frac{1}{1-\frac{a}{\rho}}]].$$

The coefficient of $\bar{r}$ is equal to $a^{-1}\,n\ln[\frac{1}{1-\frac{a}{\rho}}]$. However, whenever $T < \frac{\rho - a}{n}$, the coefficient of $\bar{r}$ is strictly between 0 and $a^{-1}\,n\ln[\frac{a}{nT} + 1]$. The constant term is nonnegative. Thus the linear combination of $\alpha\bar{r} + \beta$ is admissible as an estimator for λ under the LINEX loss function since $R(\lambda, \alpha\bar{r} + \beta)$ is continuous.

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