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Empirical Bayes Analysis of Record Statistics Based on Linex and Quadratic Loss Functions

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Abstract—In this paper, the empirical Bayes estimate is derived for the parameter of the exponential model based on record statistics. The estimate is obtained using the squared error loss and Varian's linear-exponential (LINEX) loss functions, and is compared with the corresponding maximum likelihood and Bayes estimates. Empirical Bayes prediction bounds for future record values are also obtained. A numerical example is given to illustrate the results of prediction and a Monte Carlo simulation is used to investigate the accuracy of estimation.

Keywords—Exponential distribution, Record statistics, Empirical Bayes, Estimation, Prediction, LINEX loss, Quadratic loss, Monte Carlo simulation.

1. INTRODUCTION

The one-parameter exponential distribution (denoted by $\text{Exp}(\theta)$) has a probability density function (pdf)

$$f(x;\theta) = \theta e^{-\theta x}, \qquad x > 0, \quad \theta > 0, \tag{1}$$

and a cumulative distribution function (cdf)

$$F(x;\theta) = 1 - e^{-\theta x}. \tag{2}$$

The exponential distribution is applied in a very wide variety of statistical procedures, especially in life testing problems. Many authors have developed inference procedures for the model. For example, among others, Kulldorff [1] devoted a large part of his book to the estimation of the parameters of the exponential distribution based on completely or partially grouped data. Cohen and Helm [2] discuss modified moment estimators for the parameter of the model. Based on Type I censoring, Bayesian estimation for the parameter and reliability function of the exponential model has been studied by Sinha and Gutman [3]. Ranking and subset selection procedures for an exponential population with Type I and Type II censored data are discussed in [4]. Balasubramanian and Balakrishnan [5] considered estimation of the parameter of the model using multiple Type II censored samples. Viveros and Balakrishnan [6] obtained interval estimation of the parameter based on progressively censored samples.

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Many authors consider prediction bounds for future observations from the exponential distribution. Lawless [7] developed the prediction of future observations in terms of some order statistics and the total time in a sample from an exponential population. Dunsmore [8] obtained Bayesian prediction bounds for order statistics of the future sample from the exponential life time model. Evans and Nigm [9] obtained a Bayesian prediction interval for future observations based on Type I censored data. Dunsmore [10] considered Bayesian prediction bounds for a future record from the exponential distribution. Lingappaiah [11] obtained Bayesian prediction bounds for the range of a future sample based on ranges in the earlier samples of random sizes. Colangelo and Patel [12] constructed prediction intervals for a future sample from the exponential distribution based on ranges and waiting times. AL-Hussaini and Jaheen [13] considered Bayesian prediction bounds for the median of a future sample of even size. Jaheen [14] obtained empirical Bayes prediction for the total test time, median, and range of a future sample of arbitrary size from the exponential distribution based on Type II censored samples.

One disadvantage when using squared error loss is that it penalizes overestimation or underestimation [15]. Overestimation of a parameter can lead to more severe or less severe consequences than underestimation, or vice versa. Subsequently, the use of an asymmetrical loss function, which associates greater importance to overestimation or underestimation, can be considered for the estimation of the parameter. The LINEX loss function is defined as

$$L(\Delta) = e^{a\Delta} - a\Delta - 1, \qquad a \neq 0, \tag{3}$$

where $\Delta = \hat{\phi}(\theta) - \phi(\theta)$, the scalar estimation error, if $\phi(\theta)$ is estimated by $\hat{\phi}(\theta)$ [16]. The sign of a represents the direction and its magnitude represents the degree of symmetry. First, for $a = 1$ the LINEX loss function is quite asymmetric about zero with overestimation being more costly than underestimation. Second, if $a < 0$, $L(\Delta)$ rises exponentially when $\Delta < 0$ (underestimation) and almost linearly when $\Delta > 0$ (overestimation). For a close to zero, the LINEX is approximately squared error loss, and therefore, almost symmetric.

The posterior expectation of the LINEX loss function in (3) is

$$E_\phi\left(\hat{\phi}(\theta) - \phi(\theta)\right) = e^{a\hat{\phi}(\theta)} E_\phi\left(e^{-a\phi(\theta)}\right) - a\left(\hat{\phi}(\theta) - E_\phi(\phi(\theta))\right) - 1, \tag{4}$$

where $E_\phi(.)$ denotes posterior expectation with respect to the posterior density of ϕ. The Bayes estimator of $\phi(\theta)$ denoted by $\hat{\phi}_{\rm BL}$ of the function $\phi(\theta)$ under the LINEX loss function is the value $\hat{\phi}(\theta)$ which minimizes (4). It is

$$\hat{\phi}_{\rm BL} = -\frac{1}{a} \ln\left(E_\phi\left(e^{-a\phi(\theta)}\right)\right), \tag{5}$$

provided that $E_\phi(e^{-a\phi(\theta)})$ exists and is finite, see [17].

Chandler [18] introduced the study of record values and documented many of the basic properties of records. Record values can be viewed as order statistics from a sample whose size is determined by the values and the order of occurrence of observations. In a little over thirty years, a large number of publications devoted to records have appeared. This is possibly due to the fact that we encounter this notion frequently in daily life, especially in singling out record values from a set of others and in registering and recalling record values.

Let $X_1, X_2, \dots$ be a sequence of independent and identically distributed random variables with cdf $F(x)$ and pdf $f(x)$. Set $Y_n = \max(\min) X_1, X_2, \dots, X_n$, $n \geq 1$. We say X_j is an upper (lower) record of this sequence if $Y_j > (<) Y_{j-1}$, $j > 1$. By definition, X_1 is an upper as well as a lower record value. One can transform from upper record values to lower records by replacing the original sequence of random variables by $-X_j$, $j \geq 1$ or (if $P\ (X_i) > 0 = 1$, for all i) by $1/X_i$, $i \geq 1$; the lower record values of this sequence will correspond to the upper record values of the original sequence. The notations $X_{U(n)}$ and $X_{L(n)}$ are used for the n^{th} upper and lower records, respectively. For more details on record values, see [19–22].

In this paper, the empirical Bayes estimate is derived for the parameter of the exponential model based on record statistics. The estimate is obtained using the squared error and Varian's linear-exponential (LINEX) loss functions, and is compared with the corresponding maximum likelihood and Bayes estimates. Empirical Bayes prediction bounds for future record values are also obtained. A numerical example is given to illustrate the results of prediction, and a Monte Carlo simulation is used to investigate the accuracy of estimation.

2. ESTIMATION OF THE PARAMETER

Suppose we observe n upper record values $X_{U(1)}, X_{U(2)}, \dots, X_{U(n)}$ from the Exp(θ) distribution with pdf given by (1). The likelihood function (LF) is (see [21]) given by

$$\ell(\theta \mid \underset{\sim}{x}) = \prod_{i=1}^{n-1} H(x_i) f(x_n), \tag{6}$$

where $\underset{\sim}{x} = (x_1, x_2, \dots, x_n)$ and $H(.)$ is the hazard function corresponding to the pdf $f(.)$.

It follows, from (1), (2), and (6), that

$$\ell(\theta \mid \underset{\sim}{x}) = \theta^n e^{-x_n \theta}. \tag{7}$$

The maximum likelihood estimate (MLE) of θ can be shown to be of the form

$$\hat{\theta}_{\text{ML}} = \frac{n}{x_n}. \tag{8}$$

2.1. Bayes Estimation

For Bayesian estimation, we assume a gamma (conjugate prior) density for θ with parameters α, β, and pdf

$$g(\theta) = \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta}, \qquad \theta > 0, \quad \alpha > 0, \quad \beta > 0. \tag{9}$$

It follows, from (6) and (9), that the posterior density of θ, for a given $\underset{\sim}{x}$, is given by

$$q(\theta \mid \underset{\sim}{x}) = \frac{(\beta + x_n)^{n+\alpha}}{\Gamma(n+\alpha)} \theta^{n+\alpha-1} e^{-(\beta+x_n)\theta}, \qquad \theta > 0. \tag{10}$$

Under a squared error loss function, the Bayes estimator of θ, denoted by $\hat{\theta}_{\text{BS}}$, is the mean of the posterior distribution which can be shown to be

$$\hat{\theta}_{\text{BS}} = \frac{n+\alpha}{\beta + x_n}. \tag{11}$$

Under the LINEX loss function (3), when $\Delta = \hat{\theta} - \theta$, the Bayes estimate $\hat{\theta}_{\text{BL}}$ of θ is obtained by using (5) and (10) as

$$\hat{\theta}_{\text{BL}} = \frac{n+\alpha}{a} \ln\left(1 + \frac{a}{\beta + x_n}\right), \qquad a \neq 0. \tag{12}$$

2.2. Empirical Bayes Estimation

When the prior parameters α and β are unknown, we may use the empirical Bayes approach to get their estimates. Since the prior density (9) belongs to a parametric family with unknown parameters, such parameters are to be estimated using past samples. Applying these estimates in (11) and (12), we obtain the empirical Bayes estimates of the parameter θ based on squared error and LINEX loss functions, respectively. For more details on the empirical Bayes approach, see [23].

2.2.1. Estimation of the prior parameters

When the current (informative) sample is observed, suppose that there are available m past similar samples $X_{j,U(1)}, X_{j,U(2)}, \dots, X_{j,U(n)}$, $j = 1, 2, \dots, m$ with past realizations $\theta_1, \theta_2, \dots, \theta_m$ of the random variable θ. Each sample is assumed to be an upper record sample of size n obtained from the Exp(θ) distribution with pdf given by (1). The LF of the j^{th} sample is given by (7) with x_n being replaced by $x_{n:j}$. For a sample j, $j = 1, 2, \dots, m$, the maximum likelihood estimate of the parameter θ_j is obtained from (8) and written as

$$\hat{\theta}_j \equiv Z_j = \frac{n}{x_{n:j}}. \tag{13}$$

The pdf of $X_{n:j}$, $j = 1, 2, \dots, m$, is given, see [20], by

$$\begin{aligned} f_{X_{n:j}}(x) &= f(x) \frac{[-\ln(1 - F(x))]^{n-1}}{(n-1)!} \\ &= \frac{\theta_j^n}{\Gamma(n)} x^{n-1} e^{-\theta_j x}, \qquad x > 0, \end{aligned} \tag{14}$$

which is gamma with parameters (n, θ_j).

Therefore, the conditional pdf of Z_j for a given θ_j is obtained from (14) and is given by

$$f(z_j \mid \theta_j) = \frac{(n\theta_j)^n}{\Gamma(n)} \frac{1}{z_j^{n+1}} e^{-n\theta_j / z_j}, \qquad z_j > 0, \tag{15}$$

which is the inverted gamma with parameters $(n, n\theta_j)$.

Following Schafer and Feduccia [24] and using (9) and (15), the marginal pdf of Z_j, $j = 1, 2, \dots, m$, can be shown to be

$$\begin{aligned} f(z_j) &= \int_0^\infty f(z_j \mid \theta_j) g(\theta_j)\, d\theta_j \\ &= \frac{\beta^\alpha n^n}{B(n, \alpha)} \frac{z_j^{\alpha - 1}}{(n + \beta z_j)^{n+\alpha}}, \qquad z_j > 0. \end{aligned} \tag{16}$$

Therefore, the moments estimates of the parameters α and β may be obtained by using (16), to be of the forms

$$\hat{\alpha} = \frac{(n-1)S_1^2}{(n-2)S_2 - (n-1)S_1^2} \quad \text{and} \quad \hat{\beta} = \frac{nS_1}{(n-2)S_2 - (n-1)S_1^2}, \tag{17}$$

where

$$S_1 = \sum_{j=1}^m \frac{z_j}{m} \quad \text{and} \quad S_2 = \sum_{j=1}^m \frac{z_j^2}{m}. \tag{18}$$

Therefore, the empirical Bayes estimates of the parameter θ under the squared error and LINEX loss functions are given, respectively, by

$$\hat{\theta}_{\text{EBS}} = \frac{n + \hat{\alpha}}{\hat{\beta} + x_n}, \tag{19}$$

$$\hat{\theta}_{\text{EBL}} = \frac{n + \hat{\alpha}}{a} \ln\left(1 + \frac{a}{\hat{\beta} + x_n}\right), \qquad a \neq 0, \tag{20}$$

where $\hat{\alpha}$ and $\hat{\beta}$ are the estimates of α and β given by (17).

3. PREDICTION OF FUTURE RECORD VALUES

Based on an upper record sample of size n, $X_{U(1)}, X_{U(2)}, \ldots, X_{U(n)}$, prediction is needed for the s^{th} upper record value, $1 < n < s$. Let $Y \equiv X_{U(s)}$ be the s^{th} upper record value, $1 < n < s$. The conditional pdf of Y for given x_n is given (see [21]) by

$$h(y \mid x_n; \theta) = \frac{[v(y) - v(x_n)]^{s-n-1}}{\Gamma(s-n)} \frac{f(y;\theta)}{1 - F(x_n;\theta)}, \qquad y > x_n, \tag{21}$$

where $v(.) = -\ln(1 - F(.))$.

For the Exp(θ) distribution, with pdf given by (1), the function $h(y \mid x_n; \theta)$ is obtained by using (1), (2), and (21), and can be written as

$$h(y \mid x_n; \theta) = \frac{\theta^{s-n}}{\Gamma(s-n)} (y - x_n)^{s-n-1} e^{-\theta(y-x_n)}, \qquad y > x_n. \tag{22}$$

3.1. Bayes Prediction

When the prior parameters α and β are known, a Bayesian prediction interval for a future record value Y is obtained from the Bayes predictive density function

$$h_Y^*(y \mid \underset{\sim}{x}) = \int_\Theta h_Y(y \mid \theta) q(\theta \mid \underset{\sim}{x})\, d\theta, \tag{23}$$

where $h_Y(y \mid \theta)$ is the conditional pdf of Y for the given parameter θ and $q(\theta \mid \underset{\sim}{x})$ is the posterior density function of θ for the given informative data.

The Bayes predictive density function of the future record is obtained by substituting (10) and (22) in (23), and written as

$$h_Y^*(y \mid \underset{\sim}{x}) = \frac{(\beta + x_n)^{n+\alpha}}{B(n+\alpha, s-n)} (y - x_n)^{s-n-1} (\beta + y)^{-(s+\alpha)}, \qquad y > x_n, \tag{24}$$

where $B(.,.)$ represents the beta function.

Bayesian prediction bounds for $Y = X_{U(s)}$, given the previous data, are obtained by evaluating $\Pr(Y \geq \lambda \mid \underset{\sim}{x})$, for some positive λ. It follows, from (24), that

$$\begin{aligned} \Pr(Y \geq \lambda \mid \underset{\sim}{x}) &= \int_\lambda^\infty h_Y^*(y \mid \underset{\sim}{x})\, dy, \\ &= \frac{\mathrm{IB}_{\eta(\lambda)}(n+\alpha, s-n)}{B(n+\alpha, s-n)}, \end{aligned} \tag{25}$$

where $\mathrm{IB}_{\eta(\lambda)}(.,.)$ is the incomplete beta function (see [25]) and

$$\eta(\lambda) = \frac{\beta + x_n}{\beta + \lambda}. \tag{26}$$

It can be easily shown that $h_Y^*(y \mid \underset{\sim}{x})$ is a density function on the positive half of the real line by proving that $\Pr(Y \geq x_n \mid \underset{\sim}{x}) = 1$.

Special case

For a special case, when $s = n + 1$, (25) takes the form

$$\Pr(Y \geq \lambda_1 \mid \underset{\sim}{x}) = \left(\frac{\beta + x_n}{\beta + \lambda_1}\right)^{n+\alpha}. \tag{27}$$

A $100\tau\%$ Bayesian prediction interval for $Y \equiv X_{U(n+1)}$ is such that

$$P[\mathrm{LL}(\underset{\sim}{x}) < Y < \mathrm{UL}(\underset{\sim}{x})] = \tau, \tag{28}$$

where $\mathrm{LL}(\underset{\sim}{x})$ and $\mathrm{UL}(\underset{\sim}{x})$ are the lower and upper limits satisfying

$$\begin{aligned} \mathrm{LL}(\underset{\sim}{x}) &= (\beta + x_n)\left(\frac{(1+\tau)}{2}\right)^{-1/(n+\alpha)} - \beta, \\ \mathrm{UL}(\underset{\sim}{x}) &= (\beta + x_n)\left(\frac{(1-\tau)}{2}\right)^{-1/(n+\alpha)} - \beta. \end{aligned} \tag{29}$$

3.2. Empirical Bayes Prediction

When the prior parameters α and β are unknown, we may use the empirical Bayes approach to estimate them. Applying these estimates in (24), we obtain the empirical Bayes predictive density function $\hat{h}_Y^*(y \mid \underset{\sim}{x})$ which can be used for obtaining empirical Bayes prediction bounds of Y. For more details about Bayes and empirical Bayes prediction, see [26]. Substitution of $\hat{\alpha}$ and $\hat{\beta}$, given by (17), in (29) yields the empirical Bayes prediction bounds for the future record $Y = X_{U(n+1)}$.

4. NUMERICAL COMPUTATIONS

To illustrate the previous results, a numerical example and a Monte Carlo simulation study are presented next.

4.1. Numerical Example

Bayes and empirical Bayes prediction bounds for future upper record values are computed according to the following steps

(1) For given values of ($\alpha = 3.5$, $\beta = 1.1$), we generate $\theta = 1.136$ from the gamma prior density (9). The IMSL [27] is used in the generation of the gamma random variates.

(2) Based on the generated value θ, an upper record sample of size $n = 10$ is then generated from the density of the Exp($\theta = 1.136$) distribution defined by (1), which is considered to be the informative sample. This sample is

0.073, 0.087, 3.139, 3.528, 4.671, 4.939, 8.597, 10.289, 10.718, 11.918.

(3) Using these data, 95% Bayes prediction interval for the future upper record value $X_{U(11)}$ is computed using (29) and given by $(11.9446, 16.3233)$.

(4) For given values of α, β, and n, we generate a random sample (past data) $Z_{n:j}$, $j = 1, 2, \ldots, m$, of size $m = 20$ from the marginal density of $Z_{n:j}$, given by (16), as

22.243, 7.536, 6.765, 26.304, 8.276, 6.786, 10.224, 12.863, 4.515, 28.836,
7.703, 2.687, 8.199, 4.957, 8.695, 8.013, 14.398, 11.283, 4.484, 14.582.

The moments estimates $\hat{\alpha} = 3.7224$ and $\hat{\beta} = 0.33939$ are then computed by using (17).

(5) By using the estimates of the prior parameters $\hat{\alpha} = 3.7224$, $\hat{\beta} = 0.33939$ in (29), 95% empirical Bayes prediction interval for the future upper record value $X_{U(131)}$ is given by $(11.9409, 15.6988)$.

4.2. Monte Carlo Simulation

The ML, Bayes, and empirical Bayes estimates of the parameter θ are compared based on Monte Carlo simulation as follows.

(1) For given values of the prior parameters α and β, we generate θ from (9) and then record samples of different sizes n are generated from the exponential distribution with pdf, given by (1).

(2) The ML estimate of θ is computed from (8).

(3) For given values of a, the Bayes estimates of θ are computed from (11) and (12), based on squared error and LINEX loss functions, respectively.

(4) The empirical Bayes estimates of θ based on squared error and LINEX loss functions are computed from (19) and (20), respectively.

(5) The squared deviations $(\theta^* - \theta)^2$ are computed for different sizes n where (θ^*) stands for an estimate (ML, Bayes, or empirical Bayes) of the parameter θ.

Table 1. Estimated risk (ER) of the estimates of θ for different values of n, m, a, and 1000 repetitions. ($\alpha = 3.5$, $\beta = 1.1$.)

n	$\mathrm{ER}(\hat{\theta}_{\mathrm{ML}})$	$\mathrm{ER}(\hat{\theta}_{\mathrm{BS}})$	$\mathrm{ER}(\hat{\theta}_{\mathrm{EBS}})$		a	$\mathrm{ER}(\hat{\theta}_{\mathrm{BL}})$	$\mathrm{ER}(\hat{\theta}_{\mathrm{EBL}})$	
			$m = 15$	$m = 20$			$m = 15$	$m = 20$
					5	0.1310	0.1412	0.1405
8	0.1584	0.1301	0.1483	0.1452	10	0.1279	0.1399	0.1396
					15	0.1254	0.1392	0.1387
					5	0.1284	0.1389	0.1376
12	0.1565	0.1285	0.1433	0.1399	10	0.1251	0.1375	0.1335
					15	0.1232	0.1354	0.1302
					5	0.1214	0.1365	0.1333
15	0.1542	0.1263	0.1387	0.1321	10	0.1185	0.1357	0.1303
					15	0.1163	0.1334	0.1293

(6) The above steps are repeated 1000 times and the estimated risk (ER) is computed by averaging the squared deviations over the 1000 repetitions. The computational results are displayed in Table 1.

5. CONCLUDING REMARKS

REMARK 1. In this paper, parametric Bayes and empirical Bayes estimates for the parameter of the exponential distribution are derived based on record statistics. Tiwari and Zalkikar [28] considered Bayes and empirical Bayes estimation from record samples in the nonparametric setting.

REMARK 2. It can be shown, from (11), (12), (19), and (20), that the Bayes and empirical Bayes estimates of the parameter θ that are obtained based on the LINEX loss function tend to the corresponding estimates which are based on squared error loss when a becomes zero.

REMARK 3. It may be observed, from Table 1, that the estimated risks of the three methods of estimation are decreasing when n and m are increasing. Generally, the estimated risk of the Bayes estimate of θ is the smallest estimated risk. On the other hand, the estimated risk of the empirical Bayes estimate of θ is less than the estimated risk of the maximum likelihood estimate.

REMARK 4. Different values of the prior parameters α and β rather than those listed in Table 1 have been considered but did not change the previous conclusion.

REFERENCES

1. G. Kulldorff, *Contributions to the Theory of Estimation from Grouped and Partially Grouped Samples*, Almqvist and Wiksell, Stockholm, (1961).
2. A.C. Cohen and F.R. Helm, Estimators in the exponential distribution, *Technometrics* **15**, 415–418, (1973).
3. S.K. Sinha and I. Gutman, Bayesian inference about the reliability function for the exponential distribution, *Communications in Statistics* **5**, 471–479, (1976).
4. R.L. Berger and J.S. Kim, Ranking and subset selection procedures for exponential populations with type I and type II censored data, In *Frontiers of Modern Statistical Inference Procedures*, pp. 425–448, American Sciences Press, New York, (1985).
5. K. Balasubramanian and N. Balakrishnan, Estimation for one and two parameter exponential distributions under multiple type II censoring, *Statistische Hefte* **33**, 203–216, (1992).
6. R. Viveros and N. Balakrishnan, Interval estimation of parameters of life from progressively censored data, *Technometrics* **36**, 84–91, (1994).
7. J.F. Lawless, A prediction problem concerning samples from the exponential distribution, with application in life testing, *Technometrics* **13** (4), 725–730, (1971).
8. I.R. Dunsmore, The Bayesian predictive distribution in life testing models, *Technometrics* **16**, 455–460, (1974).
9. I.G. Evans and A.M. Nigm, Bayesian prediction for the left truncated exponential distribution, *Technometrics* **22**, 201–204, (1980).
10. I.R. Dunsmore, The future occurrence of records, *Ann. Inst. Statist. Math.* **35**, 267–277, (1983).

11. G.S. Lingappaiah, Bayes prediction in exponential life testing when sample size is a random variable, *IEEE Transactions on Reliability* **R-35** (1), 106–110, (1986).
12. L. Colangelo and J.K. Patel, Prediction intervals based on ranges and waiting times for an exponential distribution, *IEEE Transactions on Reliability* **41**, 469–472, (1992).
13. E.K. AL-Hussaini and Z.F. Jaheen, Parametric prediction bounds for the future median of the exponential distribution, *Statistics* **32**, 267–275, (1999).
14. Z.F. Jaheen, Empirical Bayes prediction under the constant hazard rate model, *Statistica* **LXI** (4), 650–658, (2001).
15. A.L. Rukhin and M.M.N. Ananda, Risk behavior of variance estimators in multivariate normal distribution, *Statistics and Probability Letters* **13**, 159–166, (1992).
16. H.R. Varian, A Bayesian approcch to real estate assessment, In *Studies in Bayesian Econometrics and Statistics in Honor of Leonard J. Savage*, pp. 195–208, North Holland, Amsterdam, (1975).
17. R. Calabria and G. Pulcini, Point estimation under asymmetric loss functions for left truncated exponential samples, *Commun. Statist. Theor. Meth.* **25** (3), 285–600, (1996).
18. K.N. Chandler, The distribution and frequency of record values, *J. Royal Statis. Society B* **14**, 220–228, (1952).
19. B.C. Arnold, N. Balakrishnan and H.N. Nagaraja, *A First Course in Order Statistics*, John Wiley and Sons, New York, (1992).
20. B.C. Arnold, N. Balakrishnan and H.N. Nagaraja, *Records*, John Wiley and Sons, New York, (1998).
21. M. Ahsanullah, *Record Statistics*, Nova Science, Commack, NY, (1995).
22. S. Gulati and W.J. Padgett, *Parametric and Nonparametric Inference from Record-Breaking Data*, Springer, (2003).
23. J.L. Maritz and T. Lwin, *Empirical Bayes Methods*, Second Edition, Chapman & Hall, London, (1989).
24. R.E. Schafer and A.J. Feduccia, Prior distributions fitted to observed reliability data, *IEEE Transactions on Reliability* **R-21**, 148–154, (1972).
25. M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions with Formulas, Graphs and Mathematical Tables*, Dover, New York, (1972).
26. J. Aitchison and I.R. Dunsmore, *Statistical Prediction Analysis*, Cambridge University Press, (1975).
27. IMSL, *Reference Manual*, IMSL, Houston, TX, (1984).
28. R. Tiwari and J. Zalkikar, Bayesian inference of survival curve from record-breaking observations: Estimation and asymptotic results, *Naval Res. Logistics* **38**, 599–609, (1991).