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# New biased estimators under the LINEX loss function

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In this paper, using the asymmetric LINEX loss function we derive the risk function of the generalized Liu estimator and almost unbiased generalized Liu estimator. We also examine the risk performance of the feasible generalized Liu estimator and feasible almost unbiased generalized Liu estimator when the LINEX loss function is used.

## 1. INTRODUCTION

In comparing risk functions of some estimators, the (symmetric) quadratic loss functions have widely been used. It is interesting to note that all studies on biased estimators use the mean square error (MSE) or equivalently, the (symmetric) quadratic loss as the basis of measuring estimators' performance. It is well known that the use of symmetric loss functions may be inappropriate in many circumstances, particularly when positive and negative errors have different consequences. Varian (1975) introduced very useful asymmetric LINEX (Linear- exponential) loss function. Since, Zellner (1986) extensively discussed the properties of the LINEX loss function, several studies on the use of the LINEX loss function have been done.

When estimating the parameter $\theta$ by $\hat{\theta}$ the loss function is given by:

$$L(\hat{\theta}) = b\left[\exp(a\Delta) - a\Delta - 1\right] \tag{1.1}$$

where $a \neq 0, b > 0$ and $\Delta = \frac{\hat{\theta}-\theta}{\theta}$ is the relative estimation error in using $\hat{\theta}$ to estimate $\theta$. Since the relative estimation error does not depend on a unit, it is often used. In our investigation we assume (without loss of generality) that $b = 1$. The sign of shape parameter $a$ reflects the direction of asymmetry- we set

$a > 0$ ($a < 0$) if over-estimation is more (less) serious than under-estimation. The magnitude of $a$ reflects the degree of asymmetry. For small values of $\mid a \mid$, $L(\hat{\theta}) \simeq \frac{1}{2} b a^2 (\frac{\hat{\theta}-\theta}{\theta})^2$ which is proportional to a squared error loss (see, Zellner (1986)). So, the LINEX loss function can be regarded as a generalization of the squared error loss function allowing for asymmetry. Numerous authors have considered the LINEX loss function in various problems of interest. Examples are Chain and Janssen (1995), Ohtani (1995), Giles and Giles (1996), Zou (1997), Parsian and Sanjari Farsipour (1999), Ohtani (1999), Wan and Krumai (1999), Takada (2000). In particular, Ohtani (1995) considered the risk of the feasible generalized ridge regression (FGRR) estimator under the LINEX loss function. Ohtani showed that FGRR estimator can strictly dominate the ordinary least squares (OLS) estimator when $a$ is positive and large. Wan (1999) examine the properties of the feasible almost unbiased generalized ridge regression (FAUGRR) estimator under the asymmetric LINEX loss function. If the value $a$ is positive, then positive estimation error is regarded as more serious than negative estimation error, and vice versa. When there is the problem of multicollinearity, one of the solutions is to use the Liu estimators proposed by Liu (1993), (see also Akdeniz and Kaçıranlar (1995)). The best choice of biasing parameters include unknown parameters, they may be replaced by their sample estimates. In this case, the Liu estimators are called the feasible Liu estimators. The exact MSE of the feasible generalized Liu estimator was derived by Akdeniz and Kaçıranlar (1995).

In the spirit of results obtained by Ohtani (1986, 1995) and Wan (1999), this article examines the properties of generalized Liu (GL) and almost unbiased generalized Liu (AUGL) estimators under the asymmetric LINEX loss function, of which quadratic loss is a special case.

In Section 2, the model and estimators are presented, and a sufficient condition for the GL estimator to dominate the OLS estimator is given. In Section 3, the risk functions of the feasible GL estimators has been derived. In Section 4, the risk function of the feasible AUGL estimator is given. The relative efficiencies of the estimators are numerically compared.

## 2. MODEL AND RISK FUNCTION

Consider the multiple linear regression model

$$y = Z\gamma + \varepsilon, \tag{2.1}$$

where $y$ is an $n \times 1$ vector of observations on the dependent variable, $\gamma$ is a $\ell \times 1$ vector of regression coefficients, $Z$ is an $n \times \ell$ matrix of full column rank of observations on nonstochastic independent variables and $\epsilon$ is an $n \times 1$ vector of normal error terms with $E(\epsilon) = 0$, $E(\epsilon\epsilon') = \sigma^2 I_n$, where $\sigma$ is constant but unknown. For purpose of analysis, we reparametrize (2.1) as

$$y = X\beta + \varepsilon \tag{2.2}$$

where $X = ZT$, $\beta = T'\gamma$, $T$ is an orthogonal matrix such that $X'X = T'Z'ZT = \Lambda$, and $\Lambda$ is a diagonal matrix with the eigenvalues of $Z'Z$ as its diagonal elements. Hereafter, we work with the reparametrized model given in (2.2). Then the OLS and the GL estimators of $\beta$ are

$$\hat{\beta} = (X'X)^{-1}X'y = \Lambda^{-1}X'y, \tag{2.3}$$

and

$$\tilde{\beta} = (X'X + I)^{-1}(X'y + D\hat{\beta})$$

$$= (\Lambda + I)^{-1}(\Lambda + D)\Lambda^{-1}X'y \tag{2.4}$$

where $D$ is an $\ell \times \ell$ diagonal matrix with positive elements $0 < d_i < 1$, $i = 1, 2, ..., \ell$. The i-th element of $\hat{\beta}$ and $\tilde{\beta}$ is

$$\hat{\beta}_i = \frac{x_i'y}{\lambda_i} \tag{2.5}$$

and

$$\tilde{\beta}_i = \frac{\lambda_i + d_i}{1 + \lambda_i}\hat{\beta}_i, \ i = 1, 2, ..., \ell \tag{2.6}$$

respectively. $x_i$ is the i-th column vector of $X$, $\lambda_i$ is the i-th diagonal element of $\Lambda$. If $d_i$ is fixed the $\tilde{\beta}_i$ has bias given by

$$bias(\tilde{\beta}_i) = E(\tilde{\beta}_i) - \beta_i = -\frac{1 - d_i}{1 + \lambda_i}\beta_i, \tag{2.7}$$

and its mean squared error

$$mse(\tilde{\beta}_i) = \frac{\sigma^2(\lambda_i + d_i)^2 + \lambda_i(1 - d_i)^2\beta_i^2}{\lambda_i(1 + \lambda_i)^2}. \tag{2.8}$$

(see, Akdeniz and Kaçıranlar (1995)). In this case $mse(\tilde{\beta}_i)$ is minimized at

$$d_{i(opt)} = \frac{\lambda_i(\beta_i^2 - \sigma^2)}{\lambda_i\beta_i^2 + \sigma^2}, i = 1, 2, ..., \ell. \tag{2.9}$$

The estimator $\tilde{\beta}_i$ is non-operational . Thus, substituting $\beta_i$ and $\sigma^2$ by their unbiased estimators $\hat{\beta}_i$ and $\hat{\sigma}^2$ we obtain the estimate of $d_i$ :

$$\hat{d}_i = \frac{\lambda_i(\hat{\beta}_i^2 - \hat{\sigma}^2)}{\lambda_i\hat{\beta}_i^2 + \hat{\sigma}^2}, i = 1, 2, ..., \ell \tag{2.10}$$

where

$$\hat{\sigma}^2 = \frac{(y - X\hat{\beta})'(y - X\hat{\beta})}{n - \ell}.$$

The asymmetric LINEX loss function of $\hat{\beta}_i$ is defined as

$$L(\hat{\beta}_i) = \exp(a\Delta_i) - a\Delta_i - 1 \tag{2.11}$$

where $\Delta_i = \frac{\hat{\beta}_i - \beta_i}{\beta_i}$. Since $\frac{\hat{\beta}_i}{\beta_i} \sim N(1, \frac{1}{\theta_i^2})$, where $\theta_i = \lambda_i^{\frac{1}{2}}\beta_i/\sigma$, the risk function of $\hat{\beta}_i$ is

$$R(\hat{\beta}_i) = E\left[L(\hat{\beta}_i)\right] = E\left\{\exp\left[a(\frac{\hat{\beta}_i}{\beta_i} - 1)\right] - a(\frac{\hat{\beta}_i}{\beta_i} - 1) - 1\right\}$$

$$= E\left\{\exp\left[a(\frac{\hat{\beta}_i}{\beta_i} - 1)\right] - 1\right\} = \exp(-a)\int_{-\infty}^{+\infty} \exp(au_i)f(u_i)du_i - 1$$

$$= -1 + \exp(\frac{a^2}{2\theta_i^2}), \tag{2.12}$$

where $u_i = \frac{\hat{\beta}_i}{\beta_i}$ and $f(u_i) = (2\pi)^{-\frac{1}{2}}\theta_i \exp\left[-\frac{1}{2}(u_i - 1)^2\theta_i^2\right]$. In a similar way, the risk function of $\tilde{\beta}_i$ is

$$R(\tilde{\beta}_i) = E\left[L(\tilde{\beta}_i)\right] = E\left\{\exp\left[a(\frac{\tilde{\beta}_i}{\beta_i} - 1)\right] - a(\frac{\tilde{\beta}_i}{\beta_i} - 1) - 1\right\}$$

$$= E\left\{\exp\left[a(\delta_i^* u_i - 1)\right]\right\} - aE(\delta_i^* u_i - 1) - 1.$$

Since

$$E\left\{\exp\left[a(\delta_i^* u_i - 1)\right]\right\} = \int_{-\infty}^{+\infty} \exp\left[a(\delta_i^* u_i - 1)\right] f(u_i)du_i$$

$$= \exp\left[\frac{a^2\delta_i^{*2}}{2\theta_i^2} + a(\delta_i^* - 1)\right]$$

we obtain

$$R(\tilde{\beta}_i) = \exp\left[\frac{a^2\delta_i^{*2}}{2\theta_i^2} + a(\delta_i^* - 1)\right] - a(\delta_i^* - 1) - 1, \tag{2.13}$$

where

$$\delta_i^* = \frac{\lambda_i + d_i}{1 + \lambda_i}.$$

The partial derivative of $R(\tilde{\beta}_i)$ with respect to $d_i$ is

$$\frac{\partial R(\tilde{\beta}_i)}{\partial d_i} = \frac{a}{1+\lambda_i}\left\{(1 + \frac{a}{\theta_i^2}\delta_i^*)\exp\left[\frac{a^2\delta_i^{*2}}{2\theta_i^2} + a(\delta_i^* - 1)\right] - 1\right\}. \tag{2.14}$$

Unfortunately, the closed form of $d_i$ which minimizes the risk can not be obtained. However, we see that when $d_i = 1$, $R(\tilde{\beta}_i) = R(\hat{\beta}_i)$, and if $a > 0$, then

$$\frac{\partial R(\tilde{\beta}_i)}{\partial d_i}\rfloor_{d_i=1} = \frac{a}{1+\lambda_i}\left\{-1 + (1 + \frac{a}{\theta_i^2})\exp(\frac{a^2}{2\theta_i^2})\right\} > 0.$$

Thus, if $a > 0$,there exist $0 < d_i^* < 1$ such that for all $d_i \in (d_i^*, 1)$, $R(\tilde{\beta}_i) < R(\hat{\beta}_i)$ holds. Since the LINEX loss function of $\tilde{\beta}_i$ can be written as

$$L(\tilde{\beta}_i) = \exp\left[a(\frac{\tilde{\beta}_i}{\beta_i} - 1)\right] - a(\frac{\tilde{\beta}_i}{\beta_i} - 1) - 1$$

$$= \sum_{j=0}^{\infty} a^j \frac{(\frac{\tilde{\beta}_i}{\beta_i} - 1)^j}{j!} - a(\frac{\tilde{\beta}_i}{\beta_i} - 1) - 1$$

$$= \sum_{j=2}^{\infty} a^j \frac{(\frac{\tilde{\beta}_i}{\beta_i} - 1)^j}{j!} \tag{2.15}$$

the first term (i.e.,$j = 2$) is a quadratic function. Thus, if we use the optimal value of $d_i$ the first term of the risk function is minimized since this value of $d_i$

minimizes the *mse*. Although this value of $d_i$ does not give minimum value of the risk function with LINEX loss, we may use $d_{i(opt)} = \frac{\lambda_i(\beta_i^2-\sigma^2)}{\lambda_i\beta_i^2+\sigma^2}$ because the closed form of the optimal value of $d_i$ cannot be obtained. When

$$d_{i(opt)} = \frac{\lambda_i(\beta_i^2-\sigma^2)}{\lambda_i\beta_i^2+\sigma^2}, \delta_i^* = \frac{\lambda_i + d_{i(opt)}}{1+\lambda_i} = \frac{\lambda_i + \frac{\lambda_i(\beta_i^2-\sigma^2)}{\lambda_i\beta_i^2+\sigma^2}}{1+\lambda_i} = \frac{\theta_i^2}{1+\theta_i^2}$$

then the difference between $R(\tilde{\beta}_i)$ and $R(\hat{\beta}_i)$ is

$$R(\tilde{\beta}_i) - R(\hat{\beta}_i) = \left\{\exp\left[\frac{a^2\delta_i^{*2}}{2\theta_i^2} + a(\delta_i^* - 1)\right] - a(\delta_i^* - 1) - \exp(\frac{a^2}{2\theta_i^2})\right\}. \quad (2.16)$$

Using methods similar to those in Ohtani (1995) we can derive $R(\tilde{\beta}_i) < R(\hat{\beta}_i)$, when $d_i = \frac{\lambda_i(\beta_i^2-\sigma^2)}{\lambda_i\beta_i^2+\sigma^2}$ and $a \geq 2$. Also, note that $\delta_i^* = \frac{\theta_i^2}{1+\theta_i^2} \to 1^-$, $R(\tilde{\beta}_i) - R(\hat{\beta}_i) \to 0^-$, as $\theta_i^2 \to \infty$.

## 3.RISK PERFORMANCE OF THE FEASIBLE GL ESTIMATOR

In the previous section we have given a sufficient condition for the GL estimator with $d_i = \frac{\lambda_i(\beta_i^2-\sigma^2)}{\lambda_i\beta_i^2+\sigma^2}$ to dominate the OLS estimator when the LINEX loss function is used . But, this biasing parameter includes the unknown parameters, $\beta_i$ and $\sigma^2$,which may be replaced by their sample estimates in a practical situation. The GL estimator with $\hat{d}_i = \frac{\lambda_i(\hat{\beta}_i^2-\hat{\sigma}^2)}{\lambda_i\hat{\beta}_i^2+\hat{\sigma}^2}$ and $\hat{\sigma}^2 = \frac{(y-X\hat{\beta})'(y-X\hat{\beta})}{n-\ell}$ is called feasible GL estimator. In this section, we examine the risk performance of the feasible GL estimator when the LINEX loss function is used.

Let us denote $z_i = \lambda_i^{\frac{1}{2}}\hat{\beta}_i/\sigma$ and $V = (n-\ell)\frac{\hat{\sigma}^2}{\sigma^2}$. Then $z_i$ and $V$ are distributed as $N(\theta_i \ , 1)$ and chi-square distribution with $\nu = n - \ell$ degrees of freedom , respectively. Making use of $z_i$ and $V$ , the feasible GL estimator of $\beta_i$ can be written as

$$\tilde{\beta}_i^* = \frac{\lambda_i + \hat{d}_i}{1+\lambda_i}\hat{\beta}_i \quad (3.1)$$

or

$$\tilde{\beta}_i^* = \frac{\lambda_i + \frac{\lambda_i(\hat{\beta}_i^2-\hat{\sigma}^2)}{\lambda_i\hat{\beta}_i^2+\hat{\sigma}^2}}{1+\lambda_i}\hat{\beta}_i$$

$$= \frac{z_i^2}{z_i^2+\frac{V}{\nu}}\hat{\beta}_i.$$

Since, $z_i = \lambda_i^{\frac{1}{2}}\hat{\beta}_i/\sigma$ and $\theta_i = \lambda_i^{\frac{1}{2}}\beta_i/\sigma$, then we may write $\hat{\beta}_i = \frac{z_i\beta_i}{\theta_i}$. Thus, we have

$$\tilde{\beta}_i^* = \frac{\frac{z_i^3\beta_i}{\theta_i}}{z_i^2 + \frac{V}{\nu}} \tag{3.2}$$

or

$$\frac{\tilde{\beta}_i^*}{\beta_i} = \frac{\frac{z_i^3}{\theta_i}}{z_i^2 + \frac{V}{\nu}} \tag{3.3}$$

The risk function of $\tilde{\beta}_i^*$ is

$$R(\tilde{\beta}_i^*) = E\left\{\exp\left[a(\frac{\tilde{\beta}_i^*}{\beta_i} - 1)\right] - a(\frac{\tilde{\beta}_i^*}{\beta_i} - 1) - 1\right\} \tag{3.4}$$

$$= E\left[\sum_{j=0}^{\infty} a^j \frac{(\frac{\tilde{\beta}_i^*}{\beta_i} - 1)^j}{j!}\right] - E\left[a(\frac{\tilde{\beta}_i^*}{\beta_i} - 1) + 1\right]$$

$$= E\left[1 + \frac{a(\frac{\tilde{\beta}_i^*}{\beta_i} - 1)}{1!}\right] + E\left[\sum_{j=2}^{\infty} a^j \frac{(\frac{\tilde{\beta}_i^*}{\beta_i} - 1)^j}{j!}\right] - E\left[a(\frac{\tilde{\beta}_i^*}{\beta_i} - 1) + 1\right]$$

$$= E\left[\sum_{j=2}^{\infty} \frac{a^j}{j!} \sum_{r=0}^{j} C(j,r)(\frac{\tilde{\beta}_i^*}{\beta_i})^r(-1)^{j-r}\right]$$

$$= \sum_{j=2}^{\infty} a^j \sum_{r=0}^{j} \frac{j!}{r!(j-r)!j!}(-1)^{j-r} E\left[(\frac{\tilde{\beta}_i^*}{\beta_i})^r\right].$$

Accordingly, the risk function of $\tilde{\beta}_i^*$ is

$$R(\tilde{\beta}_i^*) = \sum_{j=2}^{\infty} a^j \sum_{r=0}^{j} \frac{1}{r!(j-r)!}(-1)^{j-r} E\left[\frac{z_i^{3r}}{\theta_i^r(z_i^2 + \frac{V}{\nu})^r}\right]. \tag{3.5}$$

Therefore, it is seen that the risk function of the feasible GL estimator , $\tilde{\beta}_i^*$, equals to the risk function of the feasible GRR estimator which is given by Ohtani (1995). On the other hand, substituting $\beta_i^2$ and $\sigma^2$ by their unbiased estimates $\hat{\beta}_i^2 - \frac{\hat{\sigma}^2}{\lambda_i}$ and $\hat{\sigma}^2$, we obtain the estimates of $d_i$ : $\tilde{d}_i = 1 - \hat{\sigma}^2 \frac{1+\lambda_i}{\lambda_i \hat{\beta}_i^2}$, $i = 1, 2, ..., \ell$ (see, for example Liu (1993)). In this case, the feasible GL estimator of $\beta_i$ can be written as

$$\tilde{b}_i = \frac{\lambda_i + \tilde{d}_i}{1 + \lambda_i} \hat{\beta}_i,$$

or

$$\tilde{b}_i = (1 - \frac{V}{\nu z_i^2}) \frac{z_i \beta_i}{\theta_i}. \tag{3.6}$$

Thus, we have

$$\frac{\tilde{b}_i}{\beta_i} = (1 - \frac{V}{\nu z_i^2}) \frac{z_i}{\theta_i}. \tag{3.7}$$

The risk function of $\tilde{b}_i$ is

$$R(\tilde{b}_i) = \sum_{j=2}^{\infty} a^j \sum_{r=0}^{j} \frac{1}{r!(j-r)!} (-1)^{j-r} E\left[ (\frac{\tilde{b}_i}{\beta_i})^r \right] \tag{3.8}$$

or

$$R(\tilde{b}_i) = \sum_{j=2}^{\infty} a^j \sum_{r=0}^{j} \frac{1}{r!(j-r)!} (-1)^{j-r} E\left[ (1 - \frac{V}{\nu z_i^2})^r (\frac{z_i}{\theta_i})^r \right]. \tag{3.9}$$

As is shown in Appendix, the r-th moment of $\frac{\tilde{b}_i}{\beta_i}$ is given by:

(1) $r = 2p$,

$$E(\frac{\tilde{b}_i}{\beta_i})^{2p} = (\frac{\theta_i^2}{2})^{-p} \sum_{q=0}^{\infty} (\frac{\theta_i^2}{2})^q \exp(-\frac{\theta_i^2}{2}) \frac{\Gamma(p+q+\frac{v+1}{2})}{q!\Gamma(q+\frac{1}{2})\Gamma(\frac{\nu}{2})} \tag{3.10}$$

$$\times \int_0^1 f^{q-p-\frac{1}{2}} \left[ \frac{f(\nu+1)-1}{\nu} \right]^{2p} (1-f)^{\frac{\nu}{2}-1} df.$$

(2) $r = 2p+1$,

$$E(\frac{\tilde{b}_i}{\beta_i})^{2p+1} = (\frac{\theta_i^2}{2})^{-p} \sum_{q=0}^{\infty} (\frac{\theta_i^2}{2})^q \exp(-\frac{\theta_i^2}{2}) \frac{\Gamma(p+q+\frac{v+1}{2}+1)}{q!\Gamma(q+\frac{3}{2})\Gamma(\frac{\nu}{2})} \tag{3.11}$$

$$\times \int_0^1 f^{q-p-\frac{1}{2}} \left[\frac{f(\nu+1)-1}{\nu}\right]^{2p+1} (1-f)^{\frac{k}{2}-1} df.$$

The risk function can be obtained by substituting (3.10) and (3.11) in (3.8).

## 4. RISK PERFORMANCE OF THE FEASIBLE AUGL ESTIMATOR

In this section, we examine the risk performance of the feasible AUGL estimator. The AUGL estimator of $\beta$ is given by Akdeniz and Kaçıranlar (1995). The generalized Liu estimator in (2.4) can be written as

$$\tilde{\beta} = (\Lambda + I)^{-1}(\Lambda + D)\hat{\beta}. \tag{4.1}$$

It is easy to see that

$$\tilde{\beta} = \left[I - (\Lambda + I)^{-1}(I - D)\right]\hat{\beta}$$

and

$$bias(\tilde{\beta}) = -(\Lambda + I)^{-1}(I - D)\beta. \tag{4.2}$$

Thus, following Kadiyala (1984) the bias corrected generalized Liu estimator of $\beta$ is given by

$$\tilde{\beta}_{BC} = \tilde{\beta} + (\Lambda + I)^{-1}(I - D)\beta. \tag{4.3}$$

If we replace $\beta$ by the biased estimator $\tilde{\beta}$ (see, Ohtani (1986)), we have the almost unbiased generalized Liu estimator, $\tilde{\beta}^o$, is given by

$$\tilde{\beta}^o = \left[I + (\Lambda + I)^{-1}(I - D)\right]\tilde{\beta}$$

or

$$\tilde{\beta}^o = \left[I - (\Lambda + I)^{-2}(I - D)^2\right]\hat{\beta}. \tag{4.4}$$

Denoting the i-th element of $\tilde{\beta}^o$ as $\tilde{\beta}_i^o$, we have

$$\tilde{\beta}_i^o = \left[1 - (\frac{1-d_i}{1+\lambda_i})^2\right]\hat{\beta}_i. \tag{4.5}$$

It is seen that $mse(\tilde{\beta}_i^o)$ is minimized at

$$d_{i(opt)} = 1 - \sqrt{\frac{\sigma^2(1+\lambda_i)^2}{\lambda_i\beta_i^2+\sigma^2}} \tag{4.6}$$

(see, Akdeniz and Kaçıranlar (1995)). Since

$$\frac{1-d_{i(opt)}}{1+\lambda_i} = \sqrt{\frac{\sigma^2}{\lambda_i\beta_i^2+\sigma^2}}$$

and

$$\frac{1-\hat{d}_{i(opt)}}{1+\lambda_i} = \sqrt{\frac{\hat{\sigma}^2}{\lambda_i\hat{\beta}_i^2+\hat{\sigma}^2}} = \sqrt{\frac{\frac{V}{\nu}}{z_i^2+\frac{V}{\nu}}}.$$

then the feasible AUGL estimator is

$$\tilde{\beta}_i^{oo} = \left[1 - (\frac{1-\hat{d}_{i(opt)}}{1+\lambda_i})^2\right]\hat{\beta}_i = (1 - \frac{\frac{V}{\nu}}{z_i^2+\frac{V}{\nu}})\hat{\beta}_i = \frac{\beta_i z_i^3}{\theta_i(z_i^2+\frac{V}{\nu})} \tag{4.7}$$

Since

$$\frac{\tilde{\beta}_i^{oo}}{\beta_i} = \frac{\tilde{\beta}_i^*}{\beta_i} = \frac{z_i^3}{\theta_i(z_i^2+\frac{V}{\nu})} \tag{4.8}$$

it is seen that, the risk function of $\tilde{\beta}_i^{oo}$ equals to the risk function of $\tilde{\beta}_i^*$. On the other hand, using the $\hat{d}_i = \frac{\lambda_i(\hat{\beta}_i^2-\hat{\sigma}^2)}{\lambda_i\hat{\beta}_i^2+\hat{\sigma}^2}$ in (4.5) and denoting $z_i = \lambda_i^{\frac{1}{2}}\hat{\beta}/\sigma$ and $V = \nu\hat{\sigma}^2/\sigma^2$, it is readily shown that the feasible AUGL estimator can be written as

$$\bar{\beta}_i = \left[1 - (\frac{1-\frac{\lambda_i(\hat{\beta}_i^2-\hat{\sigma}^2)}{\lambda_i\hat{\beta}_i^2+\hat{\sigma}^2}}{1+\lambda_i})^2\right]\hat{\beta}_i$$

$$= \left[1 - (\frac{\frac{V}{\nu}}{z_i^2+\frac{V}{\nu}})^2\right]\hat{\beta}_i = \frac{z_i^3\beta_i(z_i^2+2\frac{V}{\nu})}{\theta_i(z_i^2+\frac{V}{\nu})^2}. \tag{4.9}$$

In this case $\bar{\beta}_i$ has the same risk function with the FAUGRR estimator, $\gamma_i^*$, which is given by Wan (1999). Therefore, we omitted the derivation of the r-th moment of $\frac{\bar{\beta}_i}{\beta_i}$.

In comparing the risks, we consider relative efficiencies defined as

$$w_1 = \frac{R(\hat{\beta}_i)}{R(\tilde{\beta}_i^*)},\ w_2 = \frac{R(\hat{\beta}_i)}{R(\tilde{b}_i)},\ w_3 = \frac{R(\hat{\beta}_i)}{R(\tilde{\beta}_i^{oo})},\ w_4 = \frac{R(\hat{\beta}_i)}{R(\bar{\beta}_i)}. \tag{4.10}$$

It is seen that, $w_1 = w_3$. $\hat{\beta}_i$ is OLS estimator, $\tilde{b}_i$ and $\tilde{\beta}_i^*$ are feasible GL estimators, $\tilde{\beta}_i^{oo}$ and $\bar{\beta}_i$ are feasible AUGL estimators. If the relative risk, $w_i$, $i = 1, 2, 3, 4$ is larger than unity, the feasible GL estimator and AUGL estimator have smaller risks than the OLS estimator. If $w_4 > w_1$ (or $w_3$) , then the feasible GLE is relatively less efficient than the AUGL estimator, and *vice versa*. From (3.8) we can see that the risk of $\tilde{b}_i$ depends on the values of $a, \nu$ and $\theta_i^2$. We evaluated the relative risk, $w_2$, for $a = -1.0$, $-0.5$, $0.01$, $0.5$, $2.0$, $3.0$, $\nu = 20$, $\theta_i^2$ = various values. For purposes of comparison, the relative efficiencies $w_1$ ($w_1 = w_3$), $w_2$ and $w_4$ are given in Tables 1(a) and 1(b). We think $\nu = 20$ is good since Ohtani (1995) and Wan (1999) used this value. So similar to Ohtani (1995) and Wan (1999) we evaluated numerically the relative efficiency $w_2 = \frac{R(\hat{\beta}_i)}{R(\tilde{b}_i)}$ . We can see from Tables 1(a) and 1(b) that $\tilde{b}_i$ is dominated by the OLS estimator over a wide region of the parameter space. In other words, $\tilde{b}_i$ is inadmissible under the LINEX loss function. At least for the cases that, we have considered, $\tilde{\beta}_i^*$ has smaller risk than $\tilde{b}_i$ over a wide range of parameter space. The feasible almost unbiased generalized Liu estimator ,$\bar{\beta}_i$ does not strictly dominate the OLS estimator. On the other hand, for large values of $\theta_i^2$ , $\tilde{\beta}_i^*$ (or $\tilde{\beta}_i^{oo}$) is dominated by the OLS estimator. For relatively large values of $a$ (say $a > 2$) $\tilde{\beta}_i^*$ (or $\tilde{\beta}_i^{oo}$) and $\bar{\beta}_i$ uniformly dominate the OLS estimator. As $\theta_i^2$ increases, the relative risks $w_1 = w_3$ and $w_4$ for $a \leq 2$ decreases, attain a minimum which is less than unity, and approaches from below. The results show that as $\theta_i^2$ increases, the relative risks $w_1 = w_3$ and $w_4$ for $a = 3$ decreases , but approaches unity from above.

TABLE 1(a) Relative efficiencies of the feasible LGLE and the feasible AUGLE to the OLSE for $\nu = 20$.

| $a =$ | -1.0 | -1.0 | -1.0 | -0.5 | -0.5 | -0.5 | 0.01 | 0.01 | 0.01 |
|---|---|---|---|---|---|---|---|---|---|
| $\theta_i^2$ | $w_1$ | $w_2$ | $w_4$ | $w_1$ | $w_2$ | $w_4$ | $w_1$ | $w_2$ | $w_4$ |
| 0.2 | 2.3749 | 0.0000 | 1.2889 | 1.9704 | 0.0000 | 1.3077 | 1.8103 | 0.0816 | 1.2529 |
| 0.4 | 1.9443 | 0.0000 | 1.2964 | 1.6866 | 0.0007 | 1.2399 | 1.6134 | 0.0876 | 1.1879 |
| 0.6 | 1.6552 | 0.0000 | 1.2373 | 1.4969 | 0.0026 | 1.1753 | 1.4676 | 0.0939 | 1.0926 |
| 0.8 | 1.4563 | 0.0000 | 1.1747 | 1.3590 | 0.0058 | 1.1214 | 1.3556 | 0.1005 | 1.0572 |
| 1.0 | 1.3125 | 0.0001 | 1.1195 | 1.2543 | 0.0098 | 1.0768 | 1.2674 | 0.1075 | 0.9503 |
| 2.0 | 0.9560 | 0.0017 | 0.9507 | 0.9718 | 0.0372 | 0.9447 | 1.0150 | 0.1481 | 0.9051 |
| 3.0 | 0.8218 | 0.0082 | 0.8814 | 0.8551 | 0.0743 | 0.8904 | 0.9036 | 0.1982 | 0.8880 |
| 4.0 | 0.7606 | 0.0227 | 0.8549 | 0.7993 | 0.1221 | 0.8701 | 0.8479 | 0.2567 | 0.8851 |
| 5.0 | 0.7320 | 0.0489 | 0.8492 | 0.7722 | 01806 | 0.8665 | 0.8195 | 0.3206 | 0.9263 |
| 10.0 | 0.7371 | 0.3710 | 0.9021 | 0.7722 | 0.5017 | 0.9146 | 0.8095 | 0.5928 | 0.886 |
| 20.0 | 0.8260 | 0.6313 | 0.9719 | 0.8483 | 0.6717 | 0.9751 | 0.8713 | 0.7157 | 0.9781 |
| 30.0 | 0.8763 | 0.6345 | 0.9965 | 0.8919 | 0.6667 | 0.9927 | 0.9079 | 0.7020 | 0.9918 |

TABLE 1(b) Relative efficiencies of the feasible GLE and the feasible AUGLE to the OLSE for $\nu = 20$

| $a =$ | 0.5 | 0.5 | 0.5 | 2.0 | 2.0 | 2.0 | 3.0 | 3.0 | 3.0 |
|---|---|---|---|---|---|---|---|---|---|
| $\theta_i^2$ | $w_1$ | $w_2$ | $w_4$ | $w_1$ | $w_2$ | $w_4$ | $w_1$ | $w_2$ | $w_4$ |
| 0.2 | 1.8792 | 0.0001 | 1.1988 | 2.3422 | 0.0000 | 1.0481 | 3.2658 | 0.0000 | 1.0217 |
| 0.4 | 1.6749 | 0.0021 | 1.1577 | 2.1511 | 0.0000 | 1.0768 | 2.2953 | 0.0000 | 1.0383 |
| 0.6 | 1.5309 | 0.0086 | 1.1180 | 1.9926 | 0.0000 | 1.0851 | 2.1829 | 0.0000 | 1.0550 |
| 0.8 | 1.4208 | 0.0191 | 1.0838 | 1.8586 | 0.0000 | 1.0813 | 2.0845 | 0.0000 | 1.0571 |
| 1.0 | 1.3335 | 0.0326 | 1.0547 | 1.7457 | 0.0000 | 1.0719 | 1.9946 | 0.0000 | 1.0595 |
| 2.0 | 1.0792 | 0.1180 | 0.9632 | 1.3913 | 0.0000 | 1.0169 | 1.6422 | 0.0000 | 1.0397 |
| 3.0 | 0.9631 | 0.2106 | 0.9226 | 1.2173 | 0.0009 | 0.9816 | 1.4276 | 0.0000 | 1.0126 |
| 4.0 | 0.9031 | 0.3018 | 0.9066 | 1.1203 | 0.0082 | 0.9632 | 1.2970 | 0.0001 | 0.9941 |
| 5.0 | 0.8707 | 0.3876 | 0.9031 | 1.0621 | 0.0395 | 0.9550 | 1.2139 | 0.0008 | 0.9833 |
| 10.0 | 0.8468 | 0.6667 | 0.9367 | 0.9696 | 0.8236 | 0.9635 | 1.0587 | 0.5078 | 0.9778 |
| 20.0 | 0.8935 | 0.7615 | 0.9807 | 0.9634 | 0.9303 | 0.9877 | 1.0115 | 1.0753 | 0.9916 |
| 30.0 | 0.9233 | 0.7386 | 0.9921 | 0.9713 | 0.8703 | 0.9948 | 1.0039 | 0.9794 | 0.9969 |

## APPENDIX: DERIVATION OF $E\left[\left(\frac{\tilde{b}_i}{\beta_i}\right)^r\right]$.

In this appendix, we derive $E\left[\left(\frac{\tilde{b}_i}{\beta_i}\right)^r\right]$ for $r = 2p$. Since $E\left[\left(\frac{\tilde{b}_i}{\beta_i}\right)^r\right]$ can be derived in a similar way, the derivation for $r = 2p+1$ is omitted.

Let us consider $E\left[(1-\frac{V}{\nu z_i^2})^r(\frac{z_i}{\theta_i})^r\right]$.

$$E\left[\left(\frac{\tilde{b}_i}{\beta_i}\right)^r\right] = E\left[(1-\frac{V}{\nu z_i^2})^r(\frac{z_i}{\theta_i})^r\right] = \int_0^\infty\int_{-\infty}^\infty (1-\frac{V}{\nu z_i^2})^r(\frac{z_i}{\theta_i})^r f_1(z_i)f_2(V)dz_idV,$$

where $f_1(z_i) = (2\pi)^{-\frac{1}{2}}\exp(-\frac{(z_i-\theta_i)^2}{2})$ and $f_2(V) = \left[2^{\frac{\nu}{2}}\Gamma(\frac{\nu}{2})\right]^{-1}V^{\frac{\nu}{2}-1}\exp(-\frac{V}{2})$. Since $z_i \sim N(\theta_i, 1)$ and $V \sim \chi^2_\nu$, we have from (3.7)

$$\begin{aligned} & E\left[\left(\frac{\tilde{b}_i}{\beta_i}\right)^{2p}\right] \\ = \;& K\int_0^\infty\int_{-\infty}^\infty (1-\frac{V}{\nu z_i^2})^{2p}(\frac{z_i}{\theta_i})^{2p}\exp(-\frac{(z_i-\theta_i)^2}{2})V^{\frac{\nu}{2}-1}\exp(-\frac{V}{2})dz_idV \end{aligned}$$

$$= K\int_0^\infty\int_{-\infty}^\infty z_i^{-2p}(z_i^2-\frac{V}{\nu})^{2p}V^{\frac{\nu}{2}-1}\exp(\theta_iz_i)\exp(-\frac{V+z_i^2}{2})dz_idV \qquad \text{(A-1)}$$

where

$$K = \theta_i^{-2p}\frac{\exp(-\frac{\theta_i^2}{2})}{2^{\frac{\nu+1}{2}}\pi^{\frac{1}{2}}\Gamma(\frac{\nu}{2})}.$$

Substituting $\exp(z_i\theta_i) = \sum_{m=0}^\infty \frac{z_i^m\theta_i^m}{m!}$ in (A-1), (A-1) reduces to

$$\sum_{m=0}^\infty \frac{\theta_i^m}{m!}K\int_0^\infty\int_{-\infty}^\infty z_i^{m-2p}(z_i^2-\frac{V}{\nu})^{2p}V^{\frac{\nu}{2}-1}\exp(-\frac{V+z_i^2}{2})dz_idV. \qquad \text{(A-2)}$$

When $m = 2q+1$, the value of the integral with respect to $z_i$ is zero because the integrand is an odd function of $z_i$. Thus, (A-2) reduces to

$$\sum_{q=0}^\infty \frac{\theta_i^{2q}}{(2q)!}2K\int_0^\infty\int_0^\infty z_i^{2q-2p}(z_i^2-\frac{V}{\nu})^{2p}V^{\frac{\nu}{2}-1}\exp(-\frac{V+z_i^2}{2})dz_idV. \qquad \text{(A-3)}$$

Making use of the change of variable, $w = z_i^2$, (A-3) reduces to

$$\sum_{q=0}^{\infty} C_q \int_0^{\infty}\int_0^{\infty} w^{q-p-\frac{1}{2}}(w-\frac{V}{\nu})^{2p} V^{\frac{\nu}{2}-1}\exp(-\frac{V+w}{2})dwdV, \qquad \text{(A-4)}$$

where $C_q = \frac{\theta_i^{2q}}{(2q)!}K$. Again making use of change of variables, $t_1 = \frac{\nu}{V}w$ and $t_2 = V$, we obtain the following expression,

$$\sum_{q=0}^{\infty} C_q \int_0^{\infty}\int_0^{\infty} (\frac{Vt_1}{\nu})^{q-p-\frac{1}{2}}(\frac{Vt_1}{\nu}-\frac{V}{\nu})^{2p} t_2^{\frac{\nu}{2}-1}\frac{t_2}{\nu}\exp(-\frac{V+\frac{Vt_1}{\nu}}{2})dt_1dt_2$$

$$=\sum_{q=0}^{\infty} C_q \int_0^{\infty}\int_0^{\infty} t_1^{q-p-\frac{1}{2}}(t_1-1)^{2p}\frac{1}{\nu^{q+p+\frac{1}{2}}} t_2^{q+p+\frac{\nu-1}{2}}\exp(-\frac{\nu t_2+t_1t_2}{2\nu})dt_1dt_2. \qquad \text{(A-5)}$$

Furthermore, if we make change of variables $x = \frac{\nu+t_1}{2\nu}t_2$ and $y = p+q+\frac{\nu+1}{2}$ we obtain the following expression,

$$\int_0^{\infty} t_2^{q+p+\frac{\nu-1}{2}}\exp(-\frac{\nu t_2+t_1t_2}{2})dt_2$$

$$=\int_0^{\infty}(\frac{2\nu}{\nu+t_1}x)^{y-1}\frac{2\nu}{\nu+t_1}\exp(-x)dx$$

$$=\frac{2\nu(2\nu)^{y-1}}{(t_1+\nu)(t_1+\nu)^{y-1}}\int_0^{\infty} x^{y-1}\exp(-x)dx = \frac{(2\nu)^y}{(t_1+\nu)^y}\Gamma(y).$$

Then, we have

$$\sum_{q=0}^{\infty} C_q \nu^{\frac{\nu}{2}} 2^y \Gamma(y)\int_0^{\infty} t_1^{q-p-\frac{1}{2}}(t_1-1)^{2p}(t_1+\nu)^{-y}dt_1. \qquad \text{(A-6)}$$

Further, applying the transformation $f = \frac{t_1}{t_1+\nu}$, $0 < t_1 < \infty$, (A-6) reduces to

$$\sum_{q=0}^{\infty} D_q \int_0^1 (\frac{f\nu}{1-f})^{q-p-\frac{1}{2}}(\frac{f\nu}{1-f}-1)^{2p}(\frac{f\nu}{1-f}+\nu)^{-y}\frac{\nu}{(1-f)^2}df$$

$$=\sum_{q=0}^{\infty} D_q \int_0^1 f^{q-p-\frac{1}{2}}(1-f)^{-q+p+\frac{1}{2}}\left[f(\nu+1)-1\right]^{2p}(1-f)^{y-2p}\nu^{-y}\frac{\nu}{(1-f)^2}df$$

$$= \sum_{q=0}^{\infty} D_q \nu^{-\frac{\nu}{2}} \int_0^1 f^{q-p-\frac{1}{2}} \left[\frac{f(\nu+1)-1}{\nu}\right]^{2p} (1-f)^{\frac{\nu}{2}-1} df. \qquad \text{(A-7)}$$

where $D_q = C_q \nu^{\frac{\nu}{2}} 2^y \Gamma(y)$. Noting that $(2q)!\pi^{\frac{1}{2}} = 2^{2q}\Gamma(q+\frac{1}{2})q!$, $D_q \nu^{-\frac{\nu}{2}}$ reduces to

$$(\frac{\theta_i^2}{2})^{q-p} \exp(-\frac{\theta_i^2}{2}) \frac{\Gamma(y)}{q!\Gamma(q+\frac{1}{2})\Gamma(\frac{\nu}{2})} \qquad \text{(A-8)}$$

Substituting (A-8) in (A-7), we obtain the required expression of $E\left[(1-\frac{V}{\nu z_i^2})^r(\frac{z_i}{\theta_i})^r\right]$ with $r = 2p$ in the text.

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