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On the sampling performance of an inequality pre-test estimator of the regression error variance under LINEX loss

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We consider the estimation of the error variance of a linear regression model where prior information is available in the form of an (uncertain) inequality constraint on the coefficients. Previous studies on this and other related problems use the squared error loss in comparing estimators' performance. Here, by adopting the asymmetric LINEX loss function, we derive and numerically evaluate the *exact* risks of the inequality constrained estimator and the inequality pre-test estimator which results after a preliminary test for an inequality constraint on the coefficients. The risks based on squared error loss are special cases of our results, and we draw appropriate comparisons.

1. INTRODUCTION

Often, when estimating the parameters of a regression model, *a priori* information is available in the form of inequality constraints on the regression coefficients. Since it was first discussed by Zellner (1961), the efficiency of the inequality constrained least squares (ICLS) estimator relative to the ordinary least squares (OLS) estimator has received much attention (see Judge and Yancey (1986) and the references therein). There is also a substantial literature associated with the inequality pre-test (IPT) estimator which arises when estimation is conditional upon the outcome of a preliminary test of an inequality hypothesis (*e.g.*, Judge and Yancey (1986), Yancey *et al.* (1989), Wan (1994, 1995, 1997)).

Much of the existing literature on inequality constrained and pre-test estimation emphasizes the estimators of the regression coefficients. In practice, the regression error variance is also of interest in respect of confidence interval and hypothesis test construction. Until recently, little is known about the estimators of regression error variance in models with inequality constraints. Wan (1994, 1996), among other things, evaluated the risk characteristics of the inequality constrained and pre-test estimators of the error variance under the squared error loss measure. A somewhat related problem was discussed in Ohtani (1991).

Being symmetric, the squared error loss function imposes equal penalty on over-estimation and under-estimation of the same magnitude. However, under-estimation of the error variance leads to an overstatement of the goodness of fit measure in the estimation sample, and is arguably more serious than over-estimation of the same magnitude. Yancey *et al.* (1989) also suggested that in the case of pre-test estimators based on one sided tests, negative estimation errors have greater consequences than positive estimation errors of the same magnitude, and the linear exponential (LINEX) loss function introduced by Varian (1975) is a convenient way of capturing such asymmetric effects. Since Zellner (1986) established the statistical properties of the LINEX loss function, numerous studies have considered the use of the LINEX loss in various estimation and prediction problems. Recent examples are Parsian (1990), Srivastava and Rao (1992), Giles and Giles (1993), Parsian and Farsipour (1993), Parsian *et al.* (1993), Takagi (1994), Cain and Janssen (1995), Ohtani (1995, 1999), Zou (1997), Wan (1999) and Wan and Kurumai (1999).

In this paper, we consider the estimation of the regression error variance by adopting the LINEX loss, after a pre-test for an inequality constraint on the model's coefficients. The exact risks of the estimators are derived and numerically evaluated. The numerical evidence suggests that the results established in the literature are robust only to mild departures from squared error loss, and as the degree of asymmetry increases, rather different results emerge.

2. NOTATIONS AND THE ESTIMATORS

We are concerned with estimating the parameters in the following model,

$$y = X\beta + \varepsilon \quad , \qquad \varepsilon \sim N(0, \sigma^2 I) \tag{2.1}$$

where y and ε are $n \times 1$ vectors; X is a non-stochastic $n \times k$ matrix of full column rank; β is an unknown $k \times 1$ coefficient vector. The (uncertain) prior information available to the investigator is represented by,

$$H_0 : C'\beta \geq r, \tag{2.2}$$

where C' is a $1 \times k$ known vector and r is a known scalar.

For purposes of convenience, we follow Judge and Yancey (1986) and transform (2.1) and (2.2) as,

$$y = H\theta + \varepsilon$$

and

$$H_0 : \theta_1 \geq r_0 \tag{2.3}$$

respectively, where $H = XS^{-1/2}Q'$; $\theta = QS^{1/2}\beta$; $S = X'X$; θ_1 is the first element of θ; r_0 is a positive scalar multiple of r; and Q is an orthogonal matrix such that $QS^{-1/2}C(C'S^{-1}C)^{-1}C'S^{-1/2}Q' = \begin{pmatrix} 1 & 0' \\ 0 & 0 \end{pmatrix}$.

The inequality constrained least squares (ICLS) estimator of θ is given by,

$$\theta^{**} = I_{(-\infty,r_0)}(\tilde{\theta}_1)\theta^{*} + I_{[r_0,\infty)}(\tilde{\theta}_1)\tilde{\theta} \quad ,$$

where $\tilde{\theta} = H'y$ is the unrestricted estimator of θ, $\theta^{*} = (r_0, \tilde{\theta}'_{(k-1)})'$ is the equality constrained least squares (ECLS) estimator of θ, $\tilde{\theta}_1$ and $\tilde{\theta}_{(k-1)}$ are respectively the first and remaining k-1 elements of $\tilde{\theta}$, and $I_{(.)}(u)$ is an indicator function which is 1 if u falls in the subscripted interval and 0 otherwise. The estimator of σ^2 corresponding to θ^{**} is,

$$\sigma^{**2} = I_{(-\infty,r_0)}(\tilde{\theta}_1)\sigma^{*2} + I_{[r_0,\infty)}(\tilde{\theta}_1)\tilde{\sigma}^2 \quad , \tag{2.4}$$

where $\tilde{\sigma}^2 = \tilde{e}'\tilde{e}/(v+\delta)$ and $\sigma^{*2} = e^{*\prime}e^{*}/(v+\gamma)$ are resepctively the unrestricted and equality restricted estimators of σ^2, $\tilde{e}$ and e^{*} are the respective vectors of residuals corresponding to the use of $\tilde{\theta}$ and θ^{*} of θ, and $v = n - k$. Three common component estimators correspond to fixing $\delta = \gamma = k$ for the maximum likelihood (ML) estimators; $\delta = 0$, $\gamma = 1$ for the least squares (LS) estimators; and $\delta = 2$, $\gamma = 3$ for the minimum mean squared error (MM) estimators.

Given the uncertainty of the constraint represented by (2.3), a preliminary test of $H_0 : \theta_1 \geq r_0$ *vs.* $H_1 : \theta_1 < r_0$ may be conducted using the test statistic $t = \sqrt{v}(\tilde{\theta}_1 - r_0)/(\tilde{\sigma}\sqrt{v + \delta}) \sim t'_{(v,\eta^2)}$, where $\eta^2 = \tau^2/(2\sigma^2)$ is the non-centrality parameter and $\tau = r_0 - \theta_1$. Note that $\tau \leq 0$ when the inequality constraint is correct and vice-versa. We reject H_0 if $t < c$ and estimate θ and σ^2 by $\tilde{\theta}$ and $\tilde{\sigma}^2$ respectively, where $c < 0$ is the size - α critical value for the *Student's t* variate with v degrees of freedom. Alternatively, we do not reject H_0 if $t \geq c$, and use θ^{**} and σ^{**2}. This means that inequality pre-test (IPT) estimators of θ and σ^2 are,

$$\hat{\theta} = I_{(-\infty,c)}(t)\tilde{\theta} + I_{[c,\infty)}(t)\theta^{**}$$

and

$$\hat{\sigma}^2 = I_{(-\infty,c)}(t)\tilde{\sigma}^2 + I_{[c,\infty)}(t)\sigma^{**2} \tag{2.5}$$

respectively. When $c \geq 0$, the unrestricted estimators are always chosen regardless of the outcome of the pre-test. Among other things, Wan (1994) derived the risks of σ^{**2} and $\hat{\sigma}^2$ associated with the ML family of component estimators under squared error loss. Wan (1996) generalized this analysis to the case where σ^{**2} and $\hat{\sigma}^2$ are based on the LS and MM families of component estimators.

The squared error loss function is symmetric with respect to positive and negative estimation errors. Zellner (1986) discussed situations where a given positive estimation error may be more serious than a given negative estimation error of the same magnitude, or vice versa. Varian (1975) introduced, in his study of real estate assessment, an asymmetric loss function known as the LINEX loss which rises approximately exponentially on one side of zero, and approximately linearly on the other side. The LINEX loss function may be written as,

$$b\left(\exp(a\nabla) - a\nabla - 1\right) \quad , \tag{2.6}$$

where $\nabla = (\bar{\sigma}^2 - \sigma^2)/\sigma^2$ denotes the estimation error in using $\bar{\sigma}^2$ to estimate σ^2, a ($\neq 0$) is a shape parameter and b (> 0). This loss function reduces to the squared error loss function for small values of $|a|$. If a is positive, then over-estimation leads to larger losses relative to under-estimation of the same magnitude, and vice versa. For a full discussion of the properties of (2.6), see Zellner (1986).

Several studies have considered different estimators of the regression variance using the LINEX loss function. For instance, Srivastava and Rao (1992) studied the risk of $\tilde{\sigma}^2$ under LINEX loss and tabulated optimal choice of δ with respect to this risk. It is found that the optimal choice of δ depends on both v, the model's degrees of freedom, and a, the asymmetric parameter of the loss function. Using the same loss, Giles and Giles (1993) considered a pre-test of a set of linear equality restrictions on the coefficients, and evaluated the risks of the associated pre-test estimators of σ^2. Contrary to the results obtained under squared error loss (*e.g.*, Ohtani (1988)), Giles and Giles showed that when $a < 0$, these pre-test estimators can be strictly dominated by the unrestricted estimator even if the critical value is chosen appropriately.

Next, we derive and evaluate the risks of σ^{**2} and $\hat{\sigma}^2$ under the LINEX loss function defined in (2.6).

3. RISKS OF THE ESTIMATORS

As it stands, the LINEX loss function depends on a, the asymmetric parameter, and the scale parameter b. As b is a factor of proportionality, without loss of generality, we set b to unity in our subsequent analysis. Now, under the stated assumptions, the risks of $\tilde{\sigma}^2$ (the unrestricted estimator), σ^{*2} (the equality restricted estimator), σ^{**2} (the inequality restricted estimator) and $\hat{\sigma}^2$ (the inequality pre-test estimator) are given by the following theorem :

Theorem I :

$$R(\tilde{\sigma}^2) = \exp(-a).\left(\frac{v+\delta}{v+\delta-2a}\right)^{v/2} + \frac{a\delta}{v+\delta} - 1 \quad , \tag{3.1}$$

provided that $v + \delta - 2a > 0$.

$$R(\sigma^{*2}) = \exp\left(\frac{a\tau^2}{\sigma^2(v+\gamma-2a)} - a \right).\left(\frac{v+\gamma}{v+\gamma-2a}\right)^{(v+1)/2} - \frac{a}{v+\gamma}(1 - \gamma + \tau^2/\sigma^2) - 1 \quad , \tag{3.2}$$

provided that $v + \gamma - 2a > 0$.

$$\begin{aligned} R(\sigma^{**2}) = {} & \exp\left(\frac{a\tau^2}{\sigma^2(v+\gamma-2a)} - a \right).\left(\frac{v+\gamma}{v+\gamma-2a}\right)^{(v+1)/2} \Pr\left(z < \frac{\tau}{\sigma}\sqrt{\frac{v+\gamma}{v+\gamma-2a}} \right) \\ & - \frac{a\tau}{\sigma(v+\gamma)\sqrt{2\pi}} \exp\left(-\tau^2/(2\sigma^2)\right) - \left(\frac{a\tau^2}{\sigma^2(v+\gamma)} + \frac{a(1-\gamma)}{v+\gamma} + 1 \right) \\ & \times \Pr\left(z < \frac{\tau}{\sigma} \right) + \Pr\left(z > \frac{\tau}{\sigma} \right).\left[\exp(-a).\left(\frac{v+\delta}{v+\delta-2a}\right)^{v/2} + \frac{a\delta}{v+\delta} \right. \\ & \left. - 1 \right] \quad , \end{aligned} \tag{3.3}$$

provided that $v + \delta - 2a > 0$ and $v + \gamma - 2a > 0$, where $z = (\tilde{\theta}_1 - \theta_1)/\sigma$ is the standard normal variable.

$$\begin{aligned} R(\hat{\sigma}^2) = {} & \Pr\left(z > \frac{\tau}{\sigma} \right).\left[\exp(-a).\left(\frac{v+\delta}{v+\delta-2a}\right)^{v/2} + \frac{a\delta}{v+\delta} - 1\right] + \frac{1}{\sqrt{2\pi}}\exp\left(-\tau^2/(2\sigma^2)\right) \\ & \times \sum_{i=0}^{\infty} (\frac{\tau}{\sigma})^i (i!)^{-1}\Gamma(\frac{i+1}{2})2^{(i-1)/2}.\left\{\exp(-a).\left(\frac{v+\gamma}{v+\gamma-2a}\right)^{(v+i+1)/2}\right. \end{aligned}$$

$$\times\, I_{c_1}\left(\frac{i+1}{2},\frac{v}{2}\right) - \frac{av(\delta-\gamma)}{(v+\gamma)(v+\delta)}\, I_{c_1}\left(\frac{i+1}{2},\frac{v+2}{2}\right) - \frac{a(i+1)}{(v+\gamma)}\, I_{c_1}\left(\frac{i+3}{2},\frac{v}{2}\right)$$

$$+ \exp(-a).\left(1 - \frac{2a}{v+\delta}\right)^{-v/2}.\left(1 - I_{c_2}\left(\frac{i+1}{2},\frac{v}{2}\right)\right) + \frac{a\delta}{v+\delta} - 1\Bigg\}, \tag{3.4}$$

provided that $c < 0$, $v + \delta - 2a > 0$ and $v + \gamma - 2a > 0$, where $I_x(.,.)$ is the incomplete Beta function, $c_1 = c^2/(v+c^2)$ and $c_2 = c^2/\left(v+c^2-2av/(v+\delta)\right)$.

The proof of $R(\tilde{\sigma}^2)$ is shown in Srivastava and Rao (1992). The risk of σ^{*2} is a special case of the corresponding result given in Giles and Giles (1993). Appendix A gives the derivation of $R(\hat{\sigma}^2)$. The risk of σ^{**2} can be derived in a similar fashion, and is available upon request from the authors.

From (3.3), (3.4) and the results given in Appendix A, we observe that as $\tau \to -\infty$, $R(\sigma^{**2})$ and $R(\hat{\sigma}^2)$ collapse to $R(\tilde{\sigma}^2)$, the risk of the unrestricted estimator. As $\tau \to \infty$, the risk of σ^{**2} increases without bound while $R(\hat{\sigma}^2)$ approaches $R(\tilde{\sigma}^2)$. With some tedious manipulations, it can also be shown that as $c \to -\infty$, the risk of $\hat{\sigma}^2$ approaches that of σ^{**2}. Conversely, as $c \to 0$, $R(\hat{\sigma}^2)$ approaches $R(\tilde{\sigma}^2)$. These are in accord with the results under squared error loss (Wan (1994, 1996)).

Wan (1996) shows that in the case of a squared error loss function, the risk of $\hat{\sigma}^2$ achieves a minimum at $c = -\sqrt{v(\gamma-\delta)/(v+\delta)}$. It is found this particular value of c also results in a stationary point for the risk of $\hat{\sigma}^2$ under LINEX loss. More specifically, we have the following theorem:

Theorem II :

$$\partial R(\hat{\sigma}^2)/\partial c = 0 \text{ when } c = 0, -\infty \text{ or } -\sqrt{v(\gamma-\delta)/(v+\delta)} \quad .$$

In other words, the risk of $\hat{\sigma}^2$ achieves a stationary point at $c = 0, -\infty$ and at $c = -\sqrt{v(\gamma-\delta)/(v+\delta)}$, *i.e.*, $c = 0$ (for ML), $c = -1$ (for LS) or $c = -\sqrt{v/(v+2)}$ (for MM). However, as we shall see, the latter value of c may minimize or maximize the risk of $\hat{\sigma}^2$, depending on the value of a and the component estimators. The proof of Theorem II is given in Appendix B.

In order to better understand the risk characteristics of both the ICLS and IPT estimators of σ^2, we numerically evaluate (3.1) - (3.4) using the following parameter values : n = 20, 65, 100; k = 2, 5, 10; a = -5, -3, -0.5, 0.5, 3,; and various values of c. The Gamma and Incomplete Beta functions are calculated using subroutines from *Press et al.* (1992), and the cumulative Normal distribution function is computed using routine S15ABF from the NAG library. Some representative results appear in Figures 1 - 12.

Figures 1, 2 and 3 respectively illustrate the risks of the estimators corresponding to the ML, LS and MM components when a = -0.5 and v = 15. As expected, the results are qualitatively indistinguishable from their counterparts under squared error loss (Wan (1994, 1996)). Specifically, there is a class of pre-test estimators in the LS and MM families of component estimators such that $\hat{\sigma}^2_{LS}$ and $\hat{\sigma}^2_{MM}$ simultaneously dominate all other estimators over certain regions. This contrasts with the results based on the ML components, where pre-testing is never the best. With an appropriate choice of c, however, the risks of $\hat{\sigma}^2_{ML}$, $\hat{\sigma}^2_{LS}$ and $\hat{\sigma}^2_{MM}$ can be uniformly smaller than those of their corresponding unrestricted estimators, and the pre-test estimator which uses c = -1 (for LS) or $c = -\sqrt{v/(v+2)}$ (for MM) has the smallest risk in the family of pre-test estimators which strictly dominate $\tilde{\sigma}^2$. The latter feature is highlighted in Figure 4, which depicts the risk of $\hat{\sigma}^2_{LS}$ for various values of $c \in [-1, 0]$.

As the loss asymmetry increases, rather different results emerge. First, the sign of a is critical in determining the ranking of estimators. Our numerical results indicate that when using estimators from the LS or MM families of component estimators, the estimators corresponding to $c = -\sqrt{v(\gamma-\delta)/(v+\delta)}$, *i.e.*, c = -1 (for LS) or $c = -\sqrt{v/(v+2)}$ (for MM), may maximize or minimize the pre-test risk depending on the sign of a. Figures 6 and 7 show that when a < 0 and as $|a|$ becomes relatively large, for certain values of c, the pre-test estimators of the LS or MM components can be strictly dominated by the unrestricted estimator. Within this class of inequality pre-test estimators, the estimators corresponding to c = -1 (for LS) or $c = -\sqrt{v/(v+2)}$ (for MM) have the largest risks (see Figure 8). In contrast, when a > 0 and is sufficiently large, these pre-test estimators can yield risk uniformly smaller than those of all other inequality pre-test estimators in their respective families, as shown in Figures 10 to 12. It is also found that when a < (>) 0, the range of τ/σ over which we prefer unrestricted estimation to pre-testing increases (decreases) as the degree of asymmetry increases.

A somewhat contrasting feature is observed for the ML component estimators, however. When a < 0, regardless of the degree of asymmetry, $\hat{\sigma}^2_{ML}$ does not dominate both $\tilde{\sigma}^2_{ML}$ and σ^{**2}_{ML} simultaneously, and the family of $\hat{\sigma}^2_{ML}$'s which dominate $\tilde{\sigma}^2_{ML}$ when $|a|$ is small continue to do so as $|a|$ increases. In fact, for moderate levels of α, the range of τ/σ over which $\hat{\sigma}^2_{ML}$ is preferred to $\tilde{\sigma}^2_{ML}$ increases as a decreases. In contrast, when a > 0 and a sufficiently asymmetric loss function is adopted, the unrestricted estimator can have smaller risk than $\hat{\sigma}^2_{ML}$ almost uniformly, except if τ/σ is small, then $\hat{\sigma}^2_{ML}$ has a slight edge over $\tilde{\sigma}^2_{ML}$. Some of these results are illustrated in Figures 5 and 9.

It is also of interest to note that when $|a|$ is small (*i.e.*, under-estimation and over-estimation are equally undesirable and the loss function is close to being symmetric), the risk of the equality restricted estimator associated with the LS component is U-shaped, while those of the ML and MM components are moderately W-shaped. Such a behaviour is attributed to the bias and variance trade off when using the equality restricted estimators of the ML and MM components. Although biased at $\tau = 0$, these estimators have smaller variance than the corresponding LS estimator.However, when there are large losses associated with under-estimation, the risk functions of the equality restricted estimators of all three components become strongly W-shaped (Figues 5 - 7). This is because the equality restricted estimators associated with all three components are biased *upwards* when the equality constraint is false, which is somewhat desirable when under-estimation is more serious. Hence, the risks of the equality restricted estimators decrease initially as $|\tau/\sigma|$ increases from zero. However, as the violation of the equality constraint increases further,. the loss associated with the upward biased estimators begins to dominate any gains from over-estimation and the risks rise without bound. On the other hand, when over-estimation is more serious , the risk functions of σ^{*2}'s are again U-shaped (Figures 9 - 11) as there is no gain from over-estimating the error variance.

Similar arguments hold for the behaviour of the inequality restricted estimators. We observe from Figures 5 to 7 that the risks of the inequality restricted estimators first decline when the inequality contraint is violated ($\tau/\sigma > 0$) and under-estimation is considered to be more serious. Again, the upward bias of the inequality restricted estimators in the neighbourhood of $\tau/\sigma = 0$ first offsets the loss associated with over-estimation, though eventually the cost of upward bias dominates as τ/σ increases further and the risks of the inequality restricted estimators then increase without bound.

Finally, other things being equal, an increase in v, the model's degrees of freedom, reduces the risks of estimators, but qualitatively, the general pattern of results are essentially the same for all values of v.

FIGURE 1
RISK UNDER LINEX LOSS FOR ML COMPONENTS
(a=−0.5, v=15)

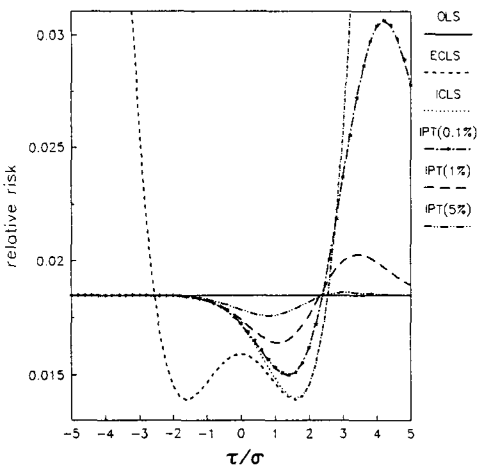


FIGURE 2
RISK UNDER LINEX LOSS FOR LS COMPONENTS
(a=−0.5, v=15)

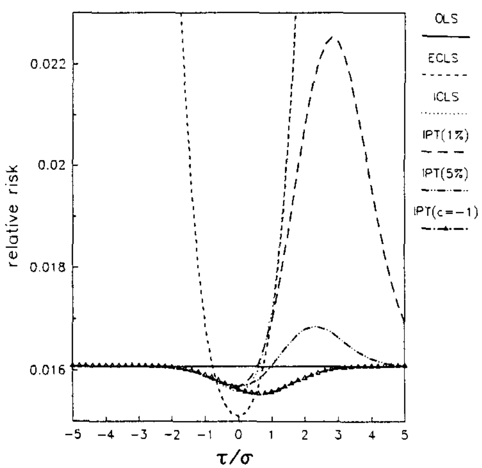


FIGURE 3
RISK UNDER LINEX LOSS FOR MM COMPONENTS
(a=−0.5, v=15)

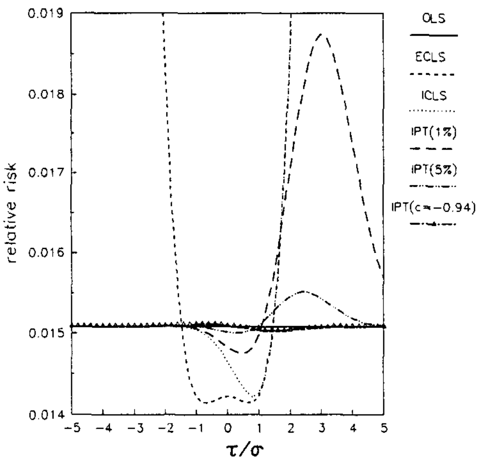


FIGURE 4
RISK UNDER LINEX LOSS FOR LS COMPONENTS
(a=−0.5, v=15)

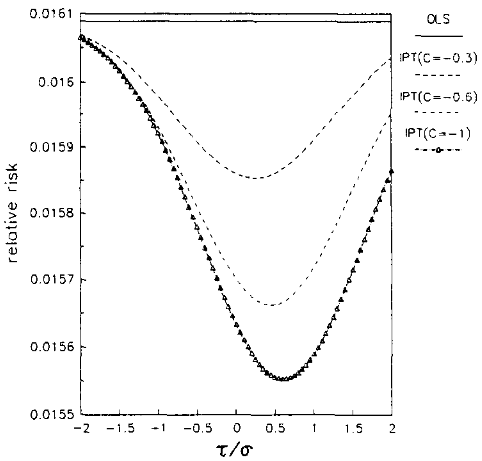


FIGURE 5
RISK UNDER LINEX LOSS FOR ML COMPONENTS
(a=−5, v=15)

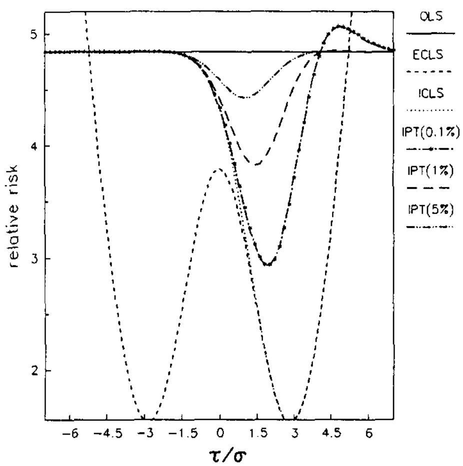


FIGURE 6
RISK UNDER LINEX LOSS FOR LS COMPONENTS
(a=−5, v=15)

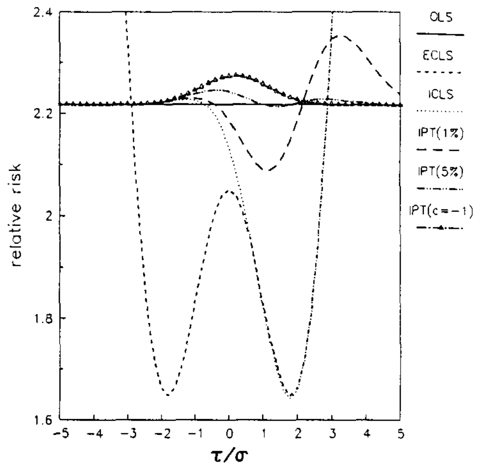


FIGURE 7
RISK UNDER LINEX LOSS FOR MM COMPONENTS
(a=−5, v=15)

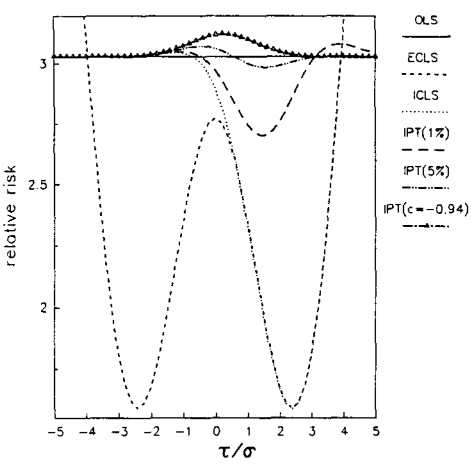


FIGURE 8
RISK UNDER LINEX LOSS FOR LS COMPONENTS
(a=−5, v=15)

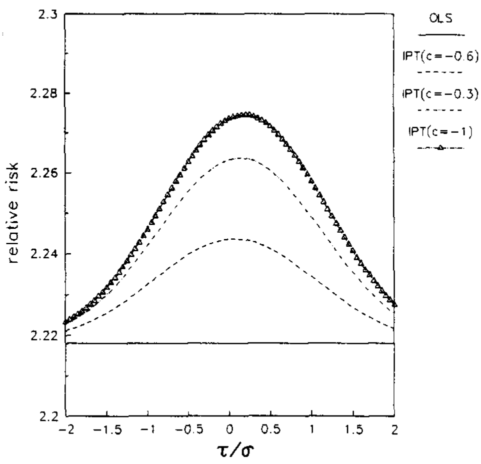


FIGURE 9
RISK UNDER LINEX LOSS FOR ML COMPONENTS
(a=3, v=15)

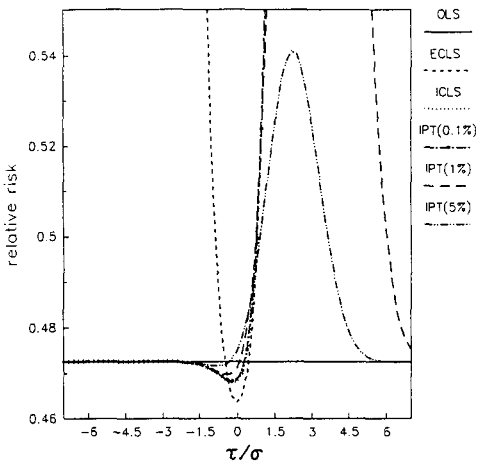


FIGURE 10
RISK UNDER LINEX LOSS FOR LS COMPONENTS
(a=3, v=15)

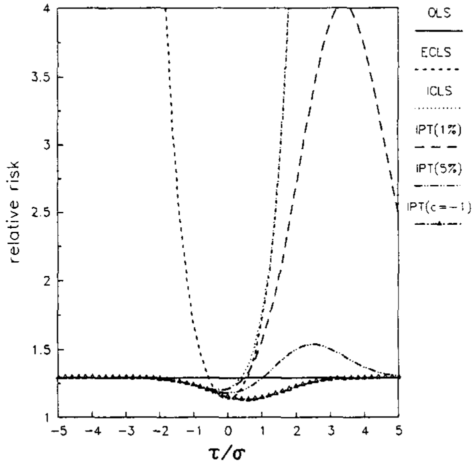


FIGURE 11
RISK UNDER LINEX LOSS FOR MM COMPONENTS
(a=3, v=15)

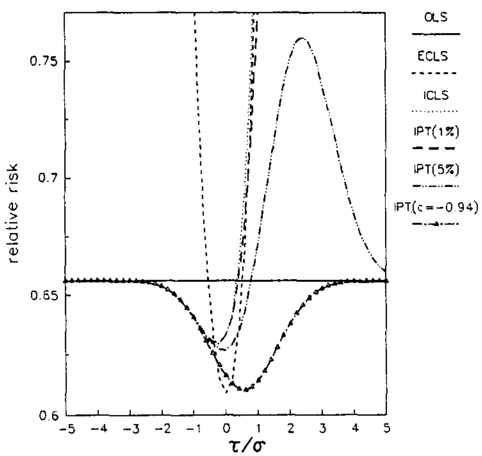
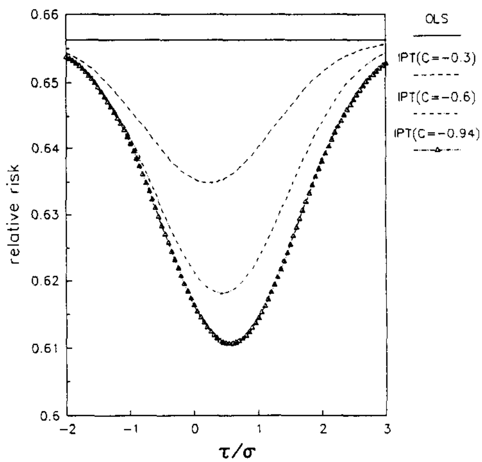


FIGURE 12
RISK UNDER LINEX LOSS FOR MM COMPONENTS
(a=3, v=15)



4. CONCLUSIONS

One broad feature that emerges from the numerical evidence is that the established results under squared error loss are only robust to mild degree of asymmetry, and any major departures from the assumed squared error loss structure change the results in a substantial manner. The decision of whether to pre-test or not also depends largely on the choice of component estimators and the level of loss asymmetry chosen by the investigator. Although the optimal choice of α is unexplored in the current paper, it suffices to say that any optimal pre-test size obtained under squared error loss (Wan (1996)) is not equivalent to that under LINEX loss. It also remains a task for future research to search for estimators which are optimal under LINEX loss. Work by Nagata (1983), Parsian and Farsipour (1993) and Parsian *et al.* (1993) may offer some insights in this regard.

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APPENDIX A : Derivation of the risk of $\hat{\sigma}^2$

Now, using the property of the indicator function, (2.5) may be written as,

$$\hat{\sigma}^2 = \tilde{\sigma}^2 + I_{[c,\infty)}(t)(\sigma^{**2} - \tilde{\sigma}^2) \qquad . \tag{A.1}$$

Using (2.4), recognizing that $e^{*\prime}e^{*} = e'e + (\sigma z - \tau)^2$, and after some manipulations, we can write,

$$\sigma^{**2} - \tilde{\sigma}^2 = I_{(-\infty,\tau/\sigma)}(z)\Big((\sigma z - \tau)^2 - \tilde{\sigma}^2(\gamma - \delta)\Big)/(v + \gamma) \qquad .$$

Hence, (A.1) can be expressed as,

$$\hat{\sigma}^2 = \tilde{\sigma}^2 + I_{[c,\infty)}(t)\Big\{I_{(-\infty,\tau/\sigma)}(z)\Big((\sigma z - \tau)^2 - \tilde{\sigma}^2(\gamma - \delta)\Big)/(v + \gamma)\Big\}.$$

Note that $I_{[c,\infty)}(t) = I_{[c',\infty)}(z)$, where $c' = c(wv^{-1})^{1/2} + \tau/\sigma$ and $w = \tilde{e}'\tilde{e}/\sigma^2$

$\sim \chi^2_v$, Also, as $c \le 0$, $I_{[c',\infty)}(z) I_{(-\infty,\tau/\sigma)}(z) = I_{(c',\tau/\sigma)}(z)$. Therefore,

$$\hat{\sigma}^2 = \tilde{\sigma}^2 + I_{[c',\tau/\sigma)}(z)\Big[(\sigma z - \tau)^2 - \tilde{\sigma}^2(\gamma - \delta)\Big]/(v + \gamma). \tag{A.2}$$

Now, writing $\tilde{\sigma}^2 = \sigma^2 w/(v+\delta)$ and applying the definition of the LINEX loss function to (A.2) with $\nabla = (\hat{\sigma}^2 - \sigma^2)/\sigma^2$ and $b = 1$, we have,

$$R(\hat{\sigma}^2) = E\Bigg\{\exp\left[a(\frac{w}{v + \delta} - 1) + I_{[c',\tau/\sigma)}(z)\left((z - \tau/\sigma)^2 - \frac{w(\gamma - \delta)}{v + \delta}\right)\frac{a}{v + \gamma}\right]$$
$$- a\left(\frac{w}{v + \delta} - 1\right) - I_{[c',\tau/\sigma)}(z)\left((z - \tau/\sigma)^2 - \frac{w(\gamma - \delta)}{v + \delta}\right)\frac{a}{v + \gamma}$$
$$- 1\Bigg\} \quad ,$$

or,

$$R(\hat{\sigma}^2) = E\Bigg\{ \int_{\tau/\sigma}^{\infty} \left[\exp\left(a(\frac{w}{v + \delta} - 1)\right) - a\left(\frac{w}{v + \delta} - 1\right) - 1\right] f(z)\, dz$$
$$+ \int_{c'}^{\tau/\sigma} \left\{\exp\left[a(\frac{w}{v + \delta} - 1) + \left((z - \tau/\sigma)^2 - \frac{w(\gamma - \delta)}{v + \delta}\right)\frac{a}{v + \gamma}\right]\right.$$
$$\left. - \left[a(\frac{w}{v + \delta} - 1) + \left((z - \tau/\sigma)^2 - \frac{w(\gamma - \delta)}{v + \delta}\right)\frac{a}{v + \gamma}\right] - 1\right\} f(z)\, dz$$
$$+ \int_{-\infty}^{c'} \left[\exp\left(a(\frac{w}{v + \delta} - 1)\right) - a\left(\frac{w}{v + \delta} - 1\right) - 1\right] f(z)\, dz \Bigg\}. \tag{A.3}$$

Using results from Srivastava and Rao (1992), we can write the first term in (A.3) as,

$$\Pr(z > \tau/\sigma)\left\{\exp(-a).\left(\frac{v+\delta}{v+\delta-2a}\right)^{v/2} + \frac{a\delta}{v+\delta} - 1\right\} \quad . \tag{A.4}$$

To consider the second term in (A.3), note that $t = (vw^{-1})^{1/2}(z - \tau/\sigma)$. Using this result and the infinite series expansion of the exponential function, we can write the second term in (A.3) as,

$$\int_c^0 \int_0^\infty \left\{\exp\left(a(\frac{w}{v+\gamma} + \frac{wt^2}{v(v+\gamma)} - 1)\right) - a(\frac{w}{v+\gamma} + \frac{wt^2}{v(v+\gamma)} - 1) - 1\right\}$$
$$\times \frac{1}{\sqrt{2\pi}} \exp\left(-\tau^2/(2\sigma^2)\right)\left(\Gamma(\tfrac{v}{2})2^{v/2}\right)^{-1} \exp\left(- \frac{w}{2} - \frac{wt^2}{2v}\right) w^{(v-1)/2} v^{-1/2}$$

$$\times \sum_{i=0}^{\infty} \left(\frac{-w^{1/2} t\tau}{v^{1/2}\sigma} \right)^i /(i!) \; dw \; dt$$

$$= \frac{1}{\sqrt{2\pi}} \exp\left[-\tau^2/(2\sigma^2)\right] \left[\Gamma(\tfrac{v}{2}) 2^{v/2}\right]^{-1} \left[(A) - (B)\right] \quad , \tag{A.5}$$

where $(A) = v^{-1/2} \int_c^0 \int_0^\infty \left[\exp\left(a(\frac{w}{v+\gamma} + \frac{wt^2}{v(v+\gamma)} - 1) - \frac{w}{2} - \frac{wt^2}{2v}\right] w^{(v-1)/2}$

$$\times \sum_{i=0}^{\infty} \left(\frac{-w^{1/2} t\tau}{v^{1/2}\sigma} \right)^i /(i!) \; dw \; dt \; ,$$

and

$$(B) = v^{-1/2} \int_c^0 \int_0^\infty \left(a(\frac{w}{v+\gamma} + \frac{wt^2}{v(v+\gamma)} - 1) + 1\right) \exp\left(- \frac{w}{2} - \frac{wt^2}{2v}\right) w^{(v-1)/2}$$

$$\times \sum_{i=0}^{\infty} \left(\frac{-w^{1/2} t\tau}{v^{1/2}\sigma} \right)^i /(i!) \; dw \; dt \quad .$$

To simplify (A), we let $\phi = \frac{v+\gamma-2a}{2v(v+\gamma)}(v+t^2)w$. Noting that $\int_0^\infty \exp(-\phi)\phi^{(v+i-1)/2} d\phi = \Gamma(\frac{v+i+1}{2})$, and after some manipulations, (A) can be written as,

$$\exp(-a)\sum_{i=0}^{\infty} v^{v/2}(-1)^i(\tau/\sigma)^i(i!)^{-1}\left(\frac{2(v+\gamma)}{v+\gamma-2a}\right)^{(v+i+1)/2} \Gamma(\frac{v+i+1}{2})$$

$$\times \int_c^0 (v + t^2)^{-(v+i+1)/2} \; t^i \; dt \quad . \tag{A.6}$$

To further simplify (A.6), we make use the change in variable $p = t^2/(v+t^2)$, and after some manipulations, we obtain,

$$\exp(-a)\sum_{i=0}^{\infty} v^{v/2}(-1)^i(\tau/\sigma)^i(i!)^{-1}\left(\frac{2(v+\gamma)}{v+\gamma-2a}\right)^{(v+i+1)/2} \Gamma(\frac{v+i+1}{2}) \int_{c_1}^0 \left(\frac{v}{1-p}\right)^{-(v+i+1)/2}$$

$$\times \left(pv/(1-p)\right)^{i/2} \frac{-v \; dp}{2(pv)^{1/2}(1-p)^{3/2}}$$

$$= \exp(-a)/2 \sum_{i=0}^{\infty} (\tau/\sigma)^i \left(\frac{2(v+\gamma)}{v+\gamma-2a}\right)^{(v+i+1)/2} \Gamma(\frac{v+i+1}{2})(i!)^{-1} \int_0^{c_1} p^{(i-1)/2}(1-p)^{v/2-1} dp$$

$$= \exp(-a)/2 \sum_{i=0}^{\infty} (\tau/\sigma)^i \left(\frac{2(v+\gamma)}{v+\gamma-2a}\right)^{(v+i+1)/2} (i!)^{-1}\Gamma(\tfrac{v}{2})\Gamma(\tfrac{i+1}{2}) I_{c_1}(\tfrac{i+1}{2},\tfrac{v}{2}) \quad , \qquad (A.7)$$

where $I_x(.,.)$ is the incomplete Beta function.

Now, to consider (B), we let $\phi = \frac{w(v+t^2)}{2v}$, and recognizing that $\int_0^{\infty} \exp(-\phi)\phi^{(v+i+1)/2}\, d\phi = \Gamma(\frac{v+i+3}{2})$ and $\int_0^{\infty} \exp(-\phi)\phi^{(v+i-1)/2}\, d\phi = \Gamma(\frac{v+i+1}{2})$, we have,

$$(B) = \sum_{i=0}^{\infty} v^{-(i+1)/2}(-1)^i(\tau/\sigma)^i(i!)^{-1}\left\{(2v)^{(v+i+3)/2}\Gamma(\frac{v+i+3}{2})\int_c^0 \left(\frac{a}{v+\gamma} + \frac{at^2}{v(v+\gamma)}\right)\right.$$

$$\times \left(v + t^2\right)^{-(v+i+3)/2} t^i\, dt - (a-1)(2v)^{(v+i+1)/2}\Gamma(\frac{v+i+1}{2})$$

$$\left.\times \int_c^0 \left(v + t^2\right)^{-(v+i+1)/2} t^i\, dt\right\} \quad .$$

Following the same procedure as above, we let $p = t^2/(v+t^2)$ and after some manipulations, we obtain,

$$(B) = \frac{1}{2}\sum_{i=0}^{\infty} (\tau/\sigma)^i(i!)^{-1}\Gamma(\tfrac{i+1}{2})\Gamma(\tfrac{v}{2})2^{(v+i+1)/2}\left(\frac{av}{v+\gamma}I_{c_1}(\frac{i+1}{2},\frac{v+2}{2}) + \frac{a(i+1)}{(v+\gamma)}\right.$$

$$\left.\times I_{c_1}(\frac{i+3}{2},\frac{v}{2}) - (a-1)I_{c_1}(\frac{i+1}{2},\frac{v}{2})\right) \quad . \qquad (A.8)$$

Substituting (A.7) and (A.8) into (A.5), we obtain,

$$\frac{1}{\sqrt{2\pi}}\exp\left(-\tau^2/(2\sigma^2)\right)\sum_{i=0}^{\infty} (\tau/\sigma)^i(i!)^{-1}\Gamma(\tfrac{i+1}{2})2^{(i-1)/2}\left\{\left[\exp(-a)\left(\frac{v+\gamma}{v+\gamma-2a}\right)^{(v+i+1)/2}\right.\right.$$

$$\left.\left. + a - 1\right] I_{c_1}(\frac{i+1}{2},\frac{v}{2}) - \frac{av}{v+\gamma}I_{c_1}(\frac{i+1}{2},\frac{v+2}{2}) - \frac{a(i+1)}{(v+\gamma)}I_{c_1}(\frac{i+3}{2},\frac{v}{2})\right\}, \qquad (A.9)$$

which is the simplified expression for the second term in (A.3).

To consider the third term in (A.3), note that,

$$E\left\{\int_{-\infty}^{c'} \left[\exp\left(a(\frac{w}{v+\delta} - 1)\right) - a(\frac{w}{v+\delta} - 1) - 1\right] f(z)\, dz\right\}$$

$$= \int_{-\infty}^{c}\int_0^{\infty} \left[\exp\left(a(\frac{w}{v+\delta} - 1)\right) - a(\frac{w}{v+\delta} - 1) - 1\right] \frac{1}{\sqrt{2\pi}} \exp\left[-\left(\frac{w^{1/2}t}{v^{1/2}} + \tau/\sigma\right)^2/2\right]$$

$$\times \frac{w^{1/2}}{v^{1/2}} \exp(-w/2)w^{v/2-1}\left(\Gamma(\tfrac{v}{2})2^{v/2}\right)^{-1} dw\, dt$$

$$= \frac{v^{-1/2}}{\sqrt{2\pi}} \exp\left(-\tau^2/(2\sigma^2)\right)\left(\Gamma(\tfrac{v}{2})2^{v/2}\right)^{-1}\int_{-\infty}^{c}\int_{0}^{\infty}\left[\exp\left(a(\frac{w}{v+\delta}-1)\right) - a(\frac{w}{v+\delta}-1)\right.$$

$$\left. - 1\right]\exp\left(-\frac{w}{2}-\frac{wt^2}{2v}\right)w^{(v-1)/2}\sum_{i=0}^{\infty}\left(\frac{-w^{1/2}t\tau}{v^{1/2}\sigma}\right)^{i}(i!)^{-1}\,dw\,dt$$

$$= \frac{v^{-1/2}}{\sqrt{2\pi}} \exp\left(-\tau^2/(2\sigma^2)\right)\left(\Gamma(\tfrac{v}{2})2^{v/2}\right)^{-1}\left((C) - (D)\right) \quad , \qquad (A.10)$$

where

$$(C) = \exp(-a)\sum_{i=0}^{\infty} v^{-i/2}(-1)^i(\tau/\sigma)^i(i!)^{-1}\int_{-\infty}^{c}\int_{0}^{\infty}\exp\left(-\frac{(v+\delta)(v+t^2)-2av}{2v(v+\delta)}\,w\right)w^{(v+i-1)/2}$$

$$\times\, t^i\,dw\,dt \quad ,$$

and

$$(D) = \sum_{i=0}^{\infty} v^{-i/2}(-1)^i(\tau/\sigma)^i(i!)^{-1}\int_{-\infty}^{c}\int_{0}^{\infty}\left(\frac{aw}{v+\delta} - a + 1\right)\exp\left(-\frac{w(v+t^2)}{2v}\right)w^{(v+i-1)/2}$$

$$\times\, t^i\,dw\,dt \quad .$$

To simplify (C), we let $\phi = \frac{(v+\delta)(v+t^2)-2av}{2v(v+\delta)}\,w$, and writing $\Gamma(\frac{v+i+1}{2}) = \int_0^{\infty} \exp(-\phi)\phi^{(v+i-1)/2}\,d\phi$, we obtain,

$$(C) = \exp(-a)\sum_{i=0}^{\infty} v^{-i/2}(-1)^i(\tau/\sigma)^i(i!)^{-1}\Gamma(\frac{v+i+1}{2})(2v)^{(v+i+1)/2}$$

$$\times \int_{-\infty}^{c}\left(v + t^2 - \frac{2av}{v+\delta}\right)^{-(v+i+1)/2} t^i\,dt \quad .$$

If we let $p = \frac{t^2}{v+t^2-\frac{2av}{v+\delta}}$ and after some manipulations, (C) can be expressed as,

$$(C) = \frac{1}{2}\exp(-a)v^{1/2}\sum_{i=0}^{\infty}(\tau/\sigma)^i(i!)^{-1}2^{(v+i+1)/2}\left(1 - \frac{2a}{v+\delta}\right)^{-v/2}\Gamma(\tfrac{i+1}{2})\Gamma(\tfrac{v}{2})$$

$$\times\left(1 - I_{c_2}(\tfrac{i+1}{2}, \tfrac{v}{2})\right) \quad . \qquad (A.11)$$

Finally, to consider (D), again, we let $\phi = \frac{w(v+t^2)}{2v}$, and obtain,

$$(D) = \sum_{i=0}^{\infty}(-1)^i(\tau/\sigma)^i(i!)^{-1}v^{-i/2}\int_{-\infty}^{c}\left[\Gamma(\frac{v+i+3}{2})\frac{a}{v+\delta}\left(\frac{2v}{v+t^2}\right)^{(v+i+3)/2} - (a-1)\right.$$

$$\times \Gamma(\tfrac{v+i+1}{2})\left(\frac{2v}{v+t^2}\right)^{(v+i+1)/2}\Bigg] t^i\, dt$$

$$= \frac{1}{2} v^{1/2} \sum_{i=0}^{\infty} (\tau/\sigma)^i (i!)^{-1} 2^{(v+i+1)/2} \Gamma(\tfrac{i+1}{2}) \Gamma(\tfrac{v}{2}) \Bigg[\frac{av}{v+\delta}\left(1 - I_{c_1}(\tfrac{i+1}{2}, \tfrac{v+2}{2})\right.$$

$$\left.- (a-1)\left(1 - I_{c_1}(\tfrac{i+1}{2}, \tfrac{v}{2})\right)\Bigg] \quad . \qquad (A.12)$$

Substituting (A.11) and (A.12) into (A.10), we obtain the following expression,

$$\frac{1}{\sqrt{2\pi}} \exp\left(-\tau^2/(2\sigma^2)\right) \sum_{i=0}^{\infty} (\tau/\sigma)^i (i!)^{-1} 2^{(i-1)/2} \Gamma(\tfrac{i+1}{2}) \Bigg[\exp(-a)\left(1 - \frac{2a}{v+\delta}\right)^{-v/2}$$

$$\times \left(1 - I_{c_2}(\tfrac{i+1}{2}, \tfrac{v}{2})\right) - \frac{av}{v+\delta}\left(1 - I_{c_1}(\tfrac{i+1}{2}, \tfrac{v+2}{2})\right) + (a-1)\left(1 - I_{c_1}(\tfrac{i+1}{2}, \tfrac{v}{2})\right)\Bigg] .$$

(A.13)

Combining (A4), (A9) and (A13), we obtain (3.4), which is the expression for the risk of the $\hat{\sigma}^2$.

APPENDIX B : Proof of $R(\hat{\sigma}^2)$ attaining a stationary point at $c = -\sqrt{v(\gamma-\delta)/(v+\delta)}$.

Now,

$$R(\hat{\sigma}^2) = E\left[\exp\left(a(\hat{\sigma}^2 - \sigma^2)/\sigma^2\right) - a(\hat{\sigma}^2 - \sigma^2)/\sigma^2 - 1\right]$$

$$= E\left\{\left[1 + a(\hat{\sigma}^2 - \sigma^2)/\sigma^2 + a^2(\hat{\sigma}^2 - \sigma^2)^2/\left((2!)\sigma^4\right)\right.\right.$$

$$\left.\left. + a^3(\hat{\sigma}^2 - \sigma^2)^3/\left((3!)\sigma^6\right) + \ldots\right] - a(\hat{\sigma}^2 - \sigma^2)/\sigma^2 - 1\right\}$$

$$= E\left[a^2(\hat{\sigma}^2 - \sigma^2)^2/\left((2!)\sigma^4\right) + a^3(\hat{\sigma}^2 - \sigma^2)^3/\left((3!)\sigma^6\right) + \ldots\ldots\right] .$$

(B.1)

From (A.2), it is easy to see that

$$(\hat{\sigma}^2 - \sigma^2)/\sigma^2 = w/(v+\delta) - 1 + I_{[c,\infty)}(t) I_{(-\infty,\tau/\sigma)}(z)\Big((z - \tau/\sigma)^2$$

$$- w(\gamma-\delta)/(v+\delta)\Big)/(v+\gamma) \qquad (B.2)$$

Substituting (B.2) into (B.1), we obtain,

$$R(\hat{\sigma}^2) = E\left\{\frac{1}{2!}\left[a(\frac{w}{v+\delta} - 1) + I_{[c,\infty)}(t) I_{(-\infty,\tau/\sigma)}(z)\Big((z - \tau/\sigma)^2 - w(\gamma-\delta)\right.\right.$$

$$/(v+\delta)\Big)\Big(a/(v+\gamma)\Big)\Big]^2 + \frac{1}{3!}\Big[a(\frac{w}{v+\delta} - 1) + I_{[c,\infty)}(t)I_{(-\infty,\tau/\sigma)}(z)$$

$$\Big((z - \tau/\sigma)^2 - w(\gamma-\delta)/(v+\delta)\Big)\Big(a/(v+\gamma)\Big)\Big]^3 + \ldots\ldots\Big\}$$

$$= E\Big\{\Pi + M\times\Omega\times I_{[c,\infty)}(t)I_{(-\infty,\tau/\sigma)}(z)\Big\} \qquad , \qquad \text{(B.3)}$$

where,

$$M = \Big((z - \tau/\sigma)^2 - w(\gamma-\delta)/(v+\delta)\Big)\Big(a/(v+\gamma)\Big) \qquad ,$$

$$\Pi = \frac{1}{2!}\Big(a(\frac{w}{v+\delta} - 1)\Big)^2 + \frac{1}{3!}\Big(a(\frac{w}{v+\delta} - 1)\Big)^3 + \ldots\ldots\ldots\ldots$$

and

$$\Omega = \frac{1}{2!}\Big(2a(\frac{w}{v+\delta} - 1) + M\Big) + \frac{1}{3!}\Big(M^2 + 3a^2(\frac{w}{v+\delta} - 1)^2 + 3Ma(\frac{w}{v+\delta} - 1)\Big) + \ldots\ldots$$

(B.3) can be simplified as,

$$R(\hat{\sigma}^2) = \int_{-\infty}^{\infty}\Big\{\int_0^{\infty}\Big(\Pi + M\times\Omega\times I_{[c,\infty)}(t)I_{(-\infty,\tau/\sigma)}(z)\Big)\, f(w)dw\Big\}\, f(z)dz$$

$$= \int_0^{\infty}\Pi\, f(w)dw + \int_0^{\infty}\Big\{\int_{-\infty}^{\tau/\sigma} M\times\Omega\times I_{[c,\infty)}(t)\, f(z)dz\Big\}\, f(w)dw$$

$$= \int_0^{\infty}\Pi\, f(w)dw + \Big\{\int_{-\infty}^{\tau/\sigma}\Big(\int_u^{\infty} M\times\Omega\times f(w)dw\Big)\, f(z)dz\Big\} \qquad ,$$

where $u = v(z - \tau/\sigma)^2/c^2$. Now,

$$\partial R(\hat{\sigma}^2)/\partial c = \int_{-\infty}^{\tau/\sigma}\Big\{\frac{\partial u}{\partial c}\times\frac{\partial}{\partial u}\Big(\int_u^{\infty} M\times\Omega\times f(w)dw\Big)\Big\}\, f(z)dz$$

$$= \int_{-\infty}^{\tau/\sigma}\Big\{\frac{2v}{c^3}(z - \tau/\sigma)^2 M^*\times\Omega^*\times f\Big(v(z - \tau/\sigma)^2/c^2\Big)\Big\}\, f(z)dz \qquad ,$$

where M^* and Ω^* are respectively the values of M and Ω when $w = v(z - \tau/\sigma)^2/c^2$.

Obviously, when $c = 0$, $f\Big(v(z - \tau/\sigma)^2/c^2\Big) = 0$, and when $c = -\infty$, $1/c^3 = 0$. Therefore, when $c = 0$ or $c = -\infty$, $\partial R(\hat{\sigma}^2)/\partial c = 0$. Moreover, as

$$M^* = \Big((z - \tau/\sigma)^2 - v(z - \tau/\sigma)^2(\gamma - \delta)/(c^2(v+\delta))\Big)a/(v+\gamma)$$

$$= (z - \tau/\sigma)^2\Big(1 - v(\gamma - \delta)/(c^2(v+\delta))\Big)a/(v+\gamma) \quad ,$$

so $M^* = 0$ when $c = -\sqrt{v(\gamma - \delta)/(v+\delta)}$.

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