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Risk performance of a pre-test estimator for normal variance with the Stein-variance estimator under the LINEX loss function

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In this paper, we derive the exact formula of the risk function of a pre-test estimator for normal variance with the Stein-variance (PTSV) estimator when the asymmetric LINEX loss function is used. Fixing the critical value of the pre-test to unity which is a suggested critical value in some sense, we examine numerically the risk performance of the PTSV estimator based on the risk function derived. Our numerical results show that although the PTSV estimator does not dominate the usual variance estimator when under-estimation is more severe than over-estimation, the PTSV estimator dominates the usual variance estimator when over-estimation is more severe. It is also shown that the dominance of the PTSV estimator over the original Stein-variance estimator is robust to the extension from the quadratic loss function to the LINEX loss function.

1 Introduction

Let x_1, x_2, ..., x_n be a random sample of size n from a normal population with mean μ and variance σ^2. When our concern is to estimate the population variance, the unbiased estimator is dominated by the

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following estimator in terms of mean squared error (MSE):

$$s^2 = \sum_{i=0}^{n}(x_i - \bar{x})^2/(n+1), \tag{1}$$

where $\bar{x}$ is the sample mean. Further, as is shown in Stein (1964), s^2 (say, the usual estimator) is dominated by the so-called Stein-variance (SV) estimator defined as

$$\widehat{\sigma}_S^2 = min[s^2,\ s_0^2], \tag{2}$$

where

$$s_0^2 = \sum_{i=0}^{n}(x_i - \mu_0)^2/(n+2), \tag{3}$$

and μ_0 is any fixed number.

When we have prior information in the form of $\sigma^2 \geq \sigma_0^2$, where σ_0^2 is some known value, it may be utilized effectively in estimation of σ^2. For example, Pandey and Mishra (1991) considered a weighted average estimator of σ_0^2 and the unbiased estimator of σ^2, and examined the sampling properties of the weighted average estimator. Also, Ohtani (1994) considered a pre-test for the null hypothesis $H_0 : \sigma^2 = \sigma_0^2$ against the alternative $H_1 : \sigma^2 > \sigma_0^2$, and examined the MSE performance of the following pre-test Stein-variance (PTSV) estimator:

$$\widehat{\sigma}^{*2} = I(J < c)\sigma_0^2 + I(J \geq c)\widehat{\sigma}_S^2, \tag{4}$$

where $I(A)$ is an indicator function such that $I(A) = 1$ if an event A occurs and $I(A) = 0$ otherwise, $J = (n+1)s^2/\sigma_0^2$ is the test statistic for H_0, and c is a critical value of the pre-test. He showed exactly that the PTSV estimator with $c = 1$ dominates the PTSV estimator with $0 \leq c < 1$ in terms of MSE. Since the PTSV estimator reduces to the original SV estimator when $c = 0$, this result shows that the PTSV estimator with $c = 1$ dominates the SV estimator.

Although the MSE has been used as a criterion in many studies on the sampling properties of shrinkage estimators, over-estimation (under-estimation) may be more serious than under-estimation (over-estimation). From this viewpoint, Varian (1975) proposed the asymmetric LINEX loss function, and Zellner (1986) discussed and analyzed intensively the statistical consequences when the LINEX loss function is used as a criterion.

The LINEX loss function for $\widehat{\sigma}^{*2}$ is defined as

$$L(\widehat{\sigma}^{*2}) = exp(a\Delta) - a\Delta - 1, \tag{5}$$

where

$$\Delta = (\widehat{\sigma}^{*2} - \sigma^2)/\sigma^2, \tag{6}$$

and a is a parameter which determines the asymmetry of $L(\sigma^{*2})$ about the origin. If the value of a is positive, then positive estimation error is regarded as more serious than negative estimation error, and vice versa. Also, when a is close to zero, the LINEX loss function is approximately the quadratic loss function. In the context of linear regression, using the LINEX loss function, Srivastava and Rao (1992) examined the risk performance of estimators for the error variance, and Giles and Giles (1993) examined the risk performance of a pre-test estimator for the error variance after a pre-test for a linear hypothesis on regression coefficients. Some other examples of the studies which use the LINEX loss function are Sadooghi-Alvandi (1990), Ohtani (1995) and Rai (1996).

In this paper, we derive the exact formula of the risk function of the PTSV estimator when the LINEX loss function is used, and we examine the risk performance of the PTSV estimator numerically based on the risk function derived. Our numerical results show that although the PTSV estimator with $c = 1$ does not dominate the usual variance estimator when a is negative, the PTSV estimator with $c = 1$ dominates the usual variance estimator when a is positive. Also, whether a is negative or positive, the PTSV estimator with $c = 1$ dominates the SV estimator. This indicates that the dominance of the PTSV estimator with $c = 1$ over the original SV estimator is robust to the extension from the quadratic loss function to the LINEX loss function.

2 Risk function

Denoting

$$u = n(\bar{x} - \mu_0)^2/\sigma^2, \tag{7}$$

$$v = \sum_{i=1}^{n}(x_i - \bar{x})^2/\sigma^2, \tag{8}$$

v is distributed as χ^2_{n-1}, and u as $\chi'^2_1(\lambda)$, where χ^2_{n-1} denotes the chi-square distribution with $n-1$ degrees of freedom, and $\chi'^2_1(\lambda)$ the noncentral chi-square distribution with 1 degree of freedom and noncentrality parameter $\lambda = n(\mu - \mu_0)^2/\sigma^2$. Noting that $\sum_{i=1}^{n}(x_i - \mu_0)^2 = \sigma^2(v + u)$ and $J = \sigma^2 v/\sigma_0^2 = v/\theta$, where $\theta = \sigma_0^2/\sigma^2 \le 1$, $\widehat{\sigma}^{*2}/\sigma^2$ is written as

$$\begin{aligned}\widehat{\sigma}^{*2}/\sigma^2 &= I(v/\theta < c)\,\theta + I(v/\theta \ge c,\ v/(v+u) < a_1/a_2)\,v/a_1 \\ &\quad + I(v/\theta \ge c,\ v/(v+u) \ge a_1/a_2)\,(v+u)/a_2, \end{aligned} \tag{9}$$

where $a_1 = n+1$ and $a_2 = n+2$.

Substituting

$$exp(a\Delta) = \sum_{k=0}^{\infty} (a\Delta)^k/k!, \tag{10}$$

in (5), the risk function of $\widehat{\sigma}^{*2}$ under LINEX loss is written as

$$\begin{aligned} R(\widehat{\sigma}^{*2}) &= E[L(\widehat{\sigma}^{*2})] \\ &= E[\sum_{k=2}^{\infty} (a^k/k!)(\widehat{\sigma}^{*2}/\sigma^2 - 1)^k]. \end{aligned} \tag{11}$$

Using the binomial expansion, $R(\widehat{\sigma}^{*2})$ reduces to

$$\begin{aligned} R(\widehat{\sigma}^{*2}) &= \sum_{k=2}^{\infty} (a^k/k!) \sum_{m=0}^{k} {}_kC_m \; E[(\widehat{\sigma}^{*2}/\sigma^2)^m](-1)^{k-m} \\ &= \sum_{k=2}^{\infty} a^k \sum_{m=0}^{k} \frac{(-1)^{k-m}}{m!(k-m)!} \; E[(\widehat{\sigma}^{*2}/\sigma^2)^m]. \end{aligned} \tag{12}$$

As is shown in Appendix, the general formula for the moments of the PTSV estimator (*i.e.*, $E[(\widehat{\sigma}^{*2}/\sigma^2)^m]$) is given by

$$\begin{aligned} & E[(\widehat{\sigma}^{*2}/\sigma^2)^m] = \theta^m P(\nu/2,\ \theta c/2) \\ & + \frac{1}{a_1^m} \sum_{i=0}^{\infty} w_i(\lambda) \frac{2^m \Gamma(\nu/2+m)}{\Gamma(\nu/2)} [1 - P(\nu/2+m,\ \theta c/2)] \\ & - \frac{1}{a_1^m} \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} w_i(\lambda) \frac{(-1)^j 2^m}{j!(1/2+i+j) a_1^{1/2+i+j}} \\ & \qquad \times \frac{\Gamma((\nu+1)/2+m+i+j)}{\Gamma(\nu/2)\Gamma(1/2+i)} \\ & \qquad \times [1 - P((\nu+1)/2+m+i+j,\ \theta c/2)] \\ & + \frac{1}{a_2^m} \sum_{i=0}^{\infty} w_i(\lambda) \frac{2^m}{\Gamma(\nu/2)\Gamma(1/2+i)} \sum_{r=0}^{m} {}_mC_r \\ & \qquad \times \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma((\nu+1)/2+m+i+j)}{j!(1/2+i+m-r+j) a_1^{1/2+i+m-r+j}} \\ & \qquad \times [1 - P((\nu+1)/2+m+i+j,\ \theta c/2)], \end{aligned} \tag{13}$$

where

$$w_i(\lambda) = exp(-\lambda/2)(\lambda/2)^i/i!, \tag{14}$$

and $P(\alpha, y)$ is the incomplete gamma function ratio defined as

$$P(\alpha, y) = \frac{1}{\Gamma(\alpha)} \int_0^y t^{\alpha-1}\ exp(-t)\ dt. \tag{15}$$

Substituting (13) in (12), we obtain the formula for the risk function.

3 Numerical results

Since the formula of $E[(\widehat{\sigma}^{*2}/\sigma^2)^m]$ is very complicated, theoretical analysis of the risk function is very difficult. Thus, we examine the risk performance of the PTSV estimator numerically based on the formula of the risk function derived in the previous section. The parameter values used in numerical evaluations were: n = 4, 6, 8, 10, 12, 14; a = -2, -1, -0.5, 0.001, 0.5, 1.0, 2.0; λ = 0.0, 0.1, 0.5, 1.0, 5.0, 10, 20, 50; θ = various values. Since it is shown in Ohtani (1994) that the PTSV estimator with $c = 1$ dominates the PTSV estimator with $c < 1$ under quadratic loss, the critical value of $c = 1$ may be of special interest. [The quadratic loss function corresponds intrinsically to the first term in the LINEX loss function.] Thus, we fix the value of c to unity (*i.e.*, $c = 1$) in our numerical evaluations.

The numerical evaluations were executed on an NEC personal computer, using FORTRAN code. Since the formula of $E[(\widehat{\sigma}^{*2})/\sigma^2)^m]$ includes the infinite series, we judged that the infinite series converged when the increment got less than 10^{-12}. Also, the incomplete gamma function ratio was evaluated by the subroutine program from Press et al. (1986). Typical results are shown in Tables 1, 2 and 3. Since the entries in the tables are the values of the relative risk of the PTSV estimator to the usual estimator (*i.e.*, $R(\sigma^{*2})/R(s^2)$), the PTSV estimator has the smaller risk than the usual estimator when the value of the relative risk is smaller than unity. Also, when $c = 0$, the PTSV estimator reduces to the original SV estimator. Since the incomplete gamma function ratios in (13) depend on θ and c only through $\theta c/2$, and since $P(\alpha,\ 0) = 0$, the value of the risk function for $\theta = 0$ is the same as that for $c = 0$. Thus, the values of the relative risk for $\theta = 0$ in the tables are the same as those of the original SV estimator.

From Table 1, we see that when $n = 4$, $a < 0$ and $\lambda \leq 1$, the relative risk is larger than unity around $\theta = 0$. This indicates that the PTSV estimator does not dominates the usual estimator when $a < 0$. However, comparing the maximum relative risk at $\theta = 0$ and the minimum relative risk at $\theta = 1$, the gain in risk of using the PTSV estimator instead of the usual estimator is larger than the loss. When $a > 0$, the values of the relative risk are uniformly smaller than unity. This indicates that the PTSV estimator dominates the usual estimator when $a > 0$. Also, whether a is positive or negative, the relative risk of the PTSV estimator decreases as θ increases from zero. Since the relative risk of the PTSV estimator at $\theta = 0$ is that of the original SV estimator, the PTSV estimator dominates the SV estimator. This indicates that the risk dominance of the PTSV estimator over the SV estimator is robust to the extension from quadratic loss to LINEX loss.

From Table 2, we see that when $n = 6$, the risk performance of the PTSV estimator is similar to the case of $n = 4$. However, the risk gain much reduces as n increases from 4 to 6. In particular, the risk gain is very small when $\lambda = 10$.

From Table 3, we see that when $n = 10$, $a < 0$ and $\lambda \leq 1$, the relative risk is larger than unity in the wide region of θ. Thus, the PTSV estimator does not dominates the usual estimator when $a < 0$. However, when $a > 0$, the PTSV estimator dominates the usual estimator. Also, in the same reason as the case of $n = 4$, the PTSV estimator dominates the original SV estimator under LINEX loss whether a is negative or positive. Comparing Tables 1, 2 and 3, we see that as n increases, the value of the relative risk increases, except for few cases (*e.g.*, $a = -2$, $\lambda \leq 1$ and $\theta \leq 0.2$).

When a is positive, the relative risk of the SV estimator (*i.e.*, the values for $\theta = 0$ in the tables) is smaller than unity. Thus, the SV estimator dominates the usual estimator when over-estimation is more serious than under-estimation. Since the SV estimator shrinks the usual estimator toward the origin, the estimate yielded by the SV estimator is smaller than the estimate yielded by the usual estimator. This indicates that even if severe over-estimation occurs when the usual estimator is used, it may be mitigated when the SV estimator is used. Further, the PTSV estimator shrinks the SV estimator. Thus, it seems to be reasonable that when a is positive, the SV estimator dominates the usual estimator, and the PTSV estimator further dominates the SV estimator. However, our numerical results show that even if a is negative, the PTSV estimator dominates the SV estimator. Although this phenomenon may be caused by the first term in the LINEX loss function (*i.e.*, the effect of MSE), there is no unambiguous interpretation of this phenomenon at present.

Although we do not show the results for $n \geq 12$, our numerical results show that the value of the relative risk does not change, at least, down to third decimal places even if the value of θ moves from zero to unity. Since we fix the value of c to unity and the value of θ is not larger

Table 1
Relative risks of the PTSV estimator with
$c = 1$ to the usual estimator for $n = 4$.

		a						
λ	θ	-2.000	-1.000	-.500	.001	.500	1.000	2.000
0.0	.0000	1.0328	1.0153	1.0015	.9817	.9518	.9026	.5946
	.1000	1.0264	1.0106	.9975	.9783	.9490	.9003	.5937
	.2000	1.0017	.9918	.9811	.9641	.9369	.8905	.5896
	.3000	.9595	.9578	.9509	.9375	.9140	.8714	.5816
	.4000	.9039	.9110	.9084	.8994	.8805	.8432	.5694
	.5000	.8398	.8548	.8564	.8519	.8381	.8069	.5532
	.6000	.7718	.7930	.7982	.7979	.7891	.7643	.5339
	.7000	.7039	.7294	.7374	.7406	.7364	.7180	.5123
	.8000	.6394	.6676	.6777	.6838	.6835	.6710	.4899
	.9000	.5807	.6109	.6226	.6311	.6342	.6268	.4685
	1.0000	.5299	.5623	.5757	.5863	.5925	.5897	.4508
1.0	.0000	1.0193	1.0070	.9972	.9829	.9610	.9239	.6636
	.1000	1.0129	1.0023	.9931	.9795	.9581	.9216	.6627
	.2000	.9882	.9834	.9767	.9653	.9461	.9117	.6586
	.3000	.9460	.9495	.9466	.9388	.9231	.8927	.6506
	.4000	.8905	.9028	.9042	.9007	.8897	.8645	.6384
	.5000	.8266	.8467	.8523	.8533	.8474	.8283	.6223
	.6000	.7588	.7851	.7942	.7994	.7984	.7858	.6030
	.7000	.6912	.7217	.7336	.7423	.7459	.7396	.5814
	.8000	.6270	.6602	.6741	.6856	.6932	.6927	.5590
	.9000	.5688	.6038	.6193	.6332	.6441	.6487	.5378
	1.0000	.5184	.5556	.5727	.5887	.6027	.6118	.5201
10.0	.0000	1.0000	.9995	.9991	.9984	.9972	.9948	.9500
	.1000	.9936	.9948	.9951	.9950	.9944	.9925	.9490
	.2000	.9690	.9760	.9787	.9809	.9824	.9826	.9450
	.3000	.9269	.9422	.9486	.9544	.9595	.9636	.9369
	.4000	.8716	.8956	.9063	.9164	.9261	.9355	.9248
	.5000	.8079	.8397	.8545	.8692	.8839	.8994	.9087
	.6000	.7405	.7783	.7967	.8154	.8351	.8570	.8894
	.7000	.6733	.7152	.7364	.7586	.7828	.8110	.8679
	.8000	.6096	.6541	.6773	.7022	.7304	.7643	.8457
	.9000	.5520	.5983	.6230	.6502	.6816	.7206	.8245
	1.0000	.5023	.5506	.5768	.6061	.6406	.6840	.8070

Table 2
Relative risks of the PTSV estimator with $c = 1$ to the usual estimator for $n = 6$.

λ	θ	a = -2.000	-1.000	-.500	.001	.500	1.000	2.000
0.0	.0000	1.0359	1.0161	1.0019	.9833	.9582	.9231	.7876
	.1000	1.0357	1.0159	1.0018	.9832	.9582	.9230	.7876
	.2000	1.0339	1.0147	1.0007	.9823	.9574	.9224	.7873
	.3000	1.0290	1.0110	.9975	.9795	.9551	.9205	.7862
	.4000	1.0199	1.0038	.9912	.9741	.9504	.9165	.7838
	.5000	1.0062	.9927	.9813	.9652	.9427	.9100	.7799
	.6000	.9879	.9774	.9674	.9528	.9317	.9005	.7741
	.7000	.9656	.9583	.9499	.9369	.9175	.8881	.7664
	.8000	.9403	.9362	.9294	.9181	.9005	.8733	.7569
	.9000	.9129	.9121	.9070	.8974	.8818	.8567	.7463
	1.0000	.8846	.8873	.8839	.8762	.8626	.8398	.7355
1.0	.0000	1.0211	1.0074	.9975	.9844	.9665	.9409	.8376
	.1000	1.0209	1.0073	.9974	.9843	.9664	.9409	.8375
	.2000	1.0192	1.0060	.9963	.9834	.9657	.9402	.8372
	.3000	1.0143	1.0023	.9932	.9806	.9633	.9383	.8361
	.4000	1.0052	.9952	.9869	.9752	.9586	.9344	.8338
	.5000	.9914	.9841	.9769	.9664	.9510	.9278	.8299
	.6000	.9732	.9688	.9631	.9539	.9400	.9184	.8241
	.7000	.9510	.9498	.9456	.9380	.9258	.9060	.8164
	.8000	.9257	.9277	.9252	.9193	.9088	.8912	.8069
	.9000	.8984	.9037	.9028	.8987	.8901	.8746	.7963
	1.0000	.8702	.8789	.8798	.8775	.8710	.8578	.7855
10.0	.0000	1.0000	.9995	.9992	.9986	.9979	.9966	.9891
	.1000	.9998	.9994	.9991	.9986	.9978	.9965	.9890
	.2000	.9981	.9981	.9980	.9977	.9970	.9959	.9887
	.3000	.9932	.9945	.9948	.9949	.9947	.9940	.9876
	.4000	.9841	.9874	.9886	.9895	.9900	.9901	.9853
	.5000	.9704	.9762	.9786	.9807	.9824	.9835	.9814
	.6000	.9522	.9610	.9648	.9683	.9714	.9741	.9756
	.7000	.9301	.9420	.9474	.9524	.9572	.9618	.9679
	.8000	.9049	.9200	.9270	.9337	.9403	.9469	.9584
	.9000	.8777	.8961	.9047	.9131	.9217	.9305	.9478
	1.0000	.8497	.8715	.8818	.8921	.9026	.9137	.9371

Table 3
Relative risks of the PTSV estimator with
$c = 1$ to the usual estimator for $n = 10$.

		a						
λ	θ	-2.000	-1.000	-.500	.001	.500	1.000	2.000
0.0	.0000	1.0337	1.0143	1.0019	.9869	.9689	.9466	.8825
	.1000	1.0337	1.0143	1.0019	.9869	.9689	.9466	.8825
	.2000	1.0337	1.0143	1.0019	.9869	.9689	.9466	.8825
	.3000	1.0337	1.0143	1.0019	.9869	.9689	.9466	.8825
	.4000	1.0337	1.0143	1.0018	.9869	.9688	.9465	.8825
	.5000	1.0335	1.0142	1.0018	.9868	.9688	.9465	.8825
	.6000	1.0332	1.0140	1.0016	.9867	.9686	.9464	.8824
	.7000	1.0328	1.0136	1.0012	.9864	.9684	.9461	.8823
	.8000	1.0320	1.0130	1.0007	.9859	.9680	.9458	.8820
	.9000	1.0309	1.0121	.9999	.9852	.9673	.9452	.8816
	1.0000	1.0295	1.0110	.9989	.9843	.9665	.9445	.8811
1.0	.0000	1.0199	1.0067	.9982	.9878	.9752	.9595	.9136
	.1000	1.0199	1.0067	.9982	.9878	.9752	.9595	.9136
	.2000	1.0199	1.0067	.9982	.9878	.9752	.9595	.9136
	.3000	1.0199	1.0067	.9981	.9878	.9752	.9595	.9136
	.4000	1.0198	1.0067	.9981	.9878	.9752	.9595	.9136
	.5000	1.0197	1.0065	.9980	.9877	.9751	.9595	.9135
	.6000	1.0194	1.0063	.9978	.9875	.9750	.9593	.9134
	.7000	1.0189	1.0060	.9975	.9873	.9747	.9591	.9133
	.8000	1.0182	1.0054	.9970	.9868	.9743	.9588	.9130
	.9000	1.0171	1.0045	.9962	.9861	.9737	.9582	.9127
	1.0000	1.0156	1.0033	.9952	.9852	.9729	.9575	.9122
10.0	.0000	1.0000	.9996	.9994	.9990	.9986	.9980	.9959
	.1000	1.0000	.9996	.9994	.9990	.9986	.9980	.9959
	.2000	1.0000	.9996	.9994	.9990	.9986	.9980	.9959
	.3000	1.0000	.9996	.9993	.9990	.9986	.9980	.9959
	.4000	1.0000	.9996	.9993	.9990	.9985	.9979	.9959
	.5000	.9998	.9995	.9992	.9989	.9985	.9979	.9959
	.6000	.9995	.9993	.9990	.9987	.9983	.9978	.9958
	.7000	.9991	.9989	.9987	.9984	.9981	.9975	.9956
	.8000	.9983	.9983	.9982	.9980	.9977	.9972	.9954
	.9000	.9972	.9974	.9974	.9973	.9970	.9967	.9950
	1.0000	.9958	.9963	.9964	.9964	.9962	.9959	.9945

than unity, the second parameter of the incomplete gamma function ratios in (13) (*i.e.*, $\theta c/2$) is smaller than 1/2. When the first parameter in the incomplete gamma function ratio is not small and the second parameter is small, the value of the incomplete gamma function ratio is very close to zero. For example, when $n = 14$, the values of the incomplete gamma function ratio for $\theta = 1$ and $\theta = 0.1$ are as follows: $P(6.5,\ 0.5) \approx 0.38 \times 10^{-5}$; $P(6.5,\ 0.1) \approx 0.15 \times 10^{-9}$. This is the reason that when $n = 14$, the value of the relative risk does not change down to third decimal places even if the value of θ moves from zero to unity. When $n \geq 15$, the same result appears. This indicates that when n is not small, conducting the pre-test does not improve significantly the SV estimator. In other words, conducting the pre-test is effective when the sample size is small (*e.g.*, $n \leq 10$). If we use the larger critical value than unity, the different risk performance may appear. However, when $c > 1$, there is no theoretical study on whether the PTSV estimator dominates the SV and/or usual estimators or not, even under quadratic loss. Thus, it is a remaining problem to examine the risk performance of the PTSV estimator with $c > 1$ under quadratic loss and/or LINEX loss.

Appendix

From (9), we have

$$\begin{aligned} E[(\hat{\sigma}^{*2}/\sigma^2)^m] &= \theta^m E[I(v < \theta c)] \\ &+ \frac{1}{a_1^m} E[I(v \geq \theta c,\ v/(v+u) < a_1/a_2)\ v^m] \\ &+ \frac{1}{a_2^m}\ E[I(v > \theta c,\ v/(v+u) \geq a_1/a_2)\ (v+u)^m] \\ &\equiv\ \theta^m E_1 + \frac{1}{a_1^m} E_2 + \frac{1}{a_2^m} E_3. \end{aligned} \tag{16}$$

First, we evaluate E_1 in (16). Since v is distributed as χ^2_ν, where $\nu = n - 1$, we have

$$E_1 = E[I(v < \theta c)] = \int_0^{\theta c} f_1(v)\ dv, \tag{17}$$

where

$$f_1(v) = \frac{1}{2^{\nu/2}\Gamma(\nu/2)}\ v^{\nu/2-1}\ exp(-v/2). \tag{18}$$

Making use of the change of variable, $z = v/2$, it is easy to see that

$$E_1 = P(\nu/2,\ \theta c/2), \tag{19}$$

where $P(\alpha,\ y)$ is the incomplete gamma function ratio defined in (15) in the text.

Second, we evaluate E_2 in (16). Since v is distributed as χ^2_ν, u as $\chi_1'^2(\lambda)$, and v and u are mutually independent, we have

$$\begin{aligned} E_2 &= E[I(v \geq \theta c,\ v/(v+u) < a_1/a_2)\ v^m] \\ &= \int\int_{R_1} v^m\ f_1(v) f_2(u)\ du dv, \end{aligned} \tag{20}$$

where R_1 is the region of v and u such that $\{(v,u)|v \geq \theta c,\ v/(v+u) < a_1/a_2\}$,

$$f_1(v)f_2(u) = \sum_{i=0}^{\infty} K_i\ v^{\nu/2-1}\ u^{1/2+i-1}\ exp[-(v+u)/2], \tag{21}$$

$$K_i = \frac{w_i(\lambda)}{2^{(\nu+1)/2+i}\Gamma(\nu/2)\Gamma(1/2+i)}, \tag{22}$$

and $w_i(\lambda)$ is given in (14) in the text. Since R_1 is equivalent to $\{(u,v)|v \geq \theta c,\ u \geq v/a_1)\}$, E_2 is rewritten as

$$\begin{aligned} E_2 &= \sum_{i=0}^{\infty} K_i \int_{\theta c}^{\infty} \int_{v/a_1}^{\infty} v^{\nu/2-1+m}\ u^{1/2+i-1} \\ &\qquad \times\ exp[-(v+u)/2]\ du dv. \end{aligned} \tag{23}$$

Making use of the change of variable, $t = u/2$, E_2 reduces to

$$\begin{aligned} &\sum_{i=0}^{\infty} K_i\ 2^{1/2+i}\Gamma(1/2+i) \int_{\theta c}^{\infty} v^{\nu/2+m-1}\ exp(-v/2)\ dv \\ &\quad - \sum_{i=0}^{\infty} K_i\ 2^{1/2+i} \int_{\theta c}^{\infty} \gamma(1/2+i,\ v/(2a_1)) \\ &\qquad \times\ v^{\nu/2+m-1}\ exp(-v/2)\ dv, \end{aligned} \tag{24}$$

where $\gamma(\alpha,\ y)$ is the incomplete gamma function defined as

$$\int_0^y t^{\alpha-1}\ exp(-t)\ dt. \tag{25}$$

The series development of $\gamma(\alpha,\ y)$ is given by

$$\gamma(\alpha,\ y) = \sum_{j=0}^{\infty} \frac{(-1)^j y^{\alpha+j}}{(\alpha+j)j!}. \tag{26}$$

[See, for example, Abramowitz and Stegun (1972, p.262).] Using (26) in the second term in (24), and making use of the change of variable, $z = v/2$, E_2 reduces to

$$\sum_{i=0}^{\infty} K_i \, 2^{1/2+i}\Gamma(1/2+i)2^{\nu/2+m}\Gamma(\nu/2+m)$$
$$\times \, [1 \, - \, P(\nu/2+m, \, \theta c/2)]$$
$$- \sum_{i=0}^{\infty} K_i \, 2^{1/2+i} \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(1/2+i+j)(2a_1)^{1/2+i+j}}$$
$$\times \, 2^{(\nu+1)/2+m+i+j}\Gamma((\nu+1)/2+m+i+j)$$
$$\times \, [1 \, - \, P((\nu+1)/2+m+i+j, \, \theta c/2)]. \qquad (27)$$

Finally, we evaluate E_3:

$$E_3 = E[I(v \geq \theta c, \; v/(v+u) \geq a_1/a_2) \; (v+u)^m]$$
$$= \int\int_{R_2} \sum_{r=0}^{m} {}_mC_r \, v^r \, u^{m-r} \, f_1(v) f_2(u) \, dudv, \qquad (28)$$

where R_2 is the region of v and u such that $\{(v,u)|v \geq \theta c, \; v/(v+u) \geq a_1/a_2\}$. Since R_2 is equivalent to $\{(v,u)|v \geq \theta c, \; u < v/a_1\}$, E_3 is rewritten as

$$E_3 = \sum_{i=0}^{\infty} K_i \sum_{r=0}^{m} {}_mC_r \int_{\theta c}^{\infty} \int_{0}^{v/a_1} v^{\nu/2+r-1} \, u^{1/2+i+m-r-1}$$
$$\times \, exp[-(v+u)/2]. \qquad (29)$$

Making use of the change of variable, $t = u/2$, E_3 reduces to

$$\sum_{i=0}^{\infty} K_i \sum_{r=0}^{m} {}_mC_r \, 2^{1/2+i+m-r} \int_{\theta c}^{\infty} \gamma(1/2+i+m-r, \; v/(2a_1))$$
$$\times \, v_1^{\nu/2+r-1} \, exp(-v/2) \, dv. \qquad (30)$$

Again, using the series development of $\gamma(\alpha, \; y)$, and making use of the change of variable, $z = v/2$, E_3 reduces to

$$\sum_{i=0}^{\infty} K_i \sum_{r=0}^{m} {}_mC_r \sum_{j=0}^{\infty} \frac{(-1)^j}{2^j \, j!(1/2+i+m-r+j)a_1^{1/2+i+m-r+j}}$$
$$\times \, 2^{(\nu+1)/2+m+i+j}\Gamma((\nu+1)/2+m+i+j)$$
$$\times \, [1 \, - \, P((\nu+1)/2+m+i+j, \, \theta c/2)]. \qquad (31)$$

Substituting K_i in (27) and (31), and further substituting (19), (27) and (31) in (16), we obtain the formula for $E[(\hat{\sigma}^{*2}/\sigma^2)^m]$ in the text.

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