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Bayes sequential estimation for a particular exponential family of distributions under LINEX loss

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Abstract The problem of sequentially estimating an unknown distribution parameter of a particular exponential family of distributions is considered under LINEX loss function for estimation error and a cost $c > 0$ for each of an i.i.d. sequence of potential observations $X_1, X_2, \dots$ A Bayesian approach is adopted and conjugate prior distributions are assumed. Asymptotically pointwise optimal and asymptotically optimal procedures are derived.

Keywords AO rule · APO rule · Bayes sequential estimation · LINEX loss function · Transformed chi-square family of distributions

Mathematics Subject Classification (2000) 62L12 · 62F10 · 62F15

1 Introduction

The aim of the Bayes sequential estimation is to derive an optimal sequential procedure consisting of an optimal stopping rule and a Bayes estimate. Usually, obtaining the Bayes estimate is possible in the problem. Then the Bayes sequential estimation problem reduces to finding an optimal (Bayes) stopping rule.

Let $\{Y_n, n \geq 1\}$ be a sequence of random variables defined on a probability space $(\Omega, \mathcal{F}, P)$, where Y_n is $\mathcal{F}_n$ -measurable and $\mathcal{F}_n \subset \mathcal{F}_{n+1} \subset \cdots \subset \mathcal{F}$ is an increasing sequence of σ-fields. Define $Z_n(c) = Y_n + cn$, $c > 0$. Let $\mathcal{N}$ be the class of all stopping times N with respect to the filtration $\{\mathcal{F}_n, n \geq 1\}$. A stopping rule $N^* = N^*(c)$

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is called a Bayes stopping rule if it satisfies

$$E(Z_{N^*}(c)) = \inf_{N \in \mathcal{N}} E(Z_N(c)).$$

The existence of Bayes solutions in sequential analysis was proved in Arrow et al. (1949), but the difficulties involved in computing them explicitly led Wald to introduce asymptotic sequential analysis in estimation (see Wald 1950). Asymptotic in his sense, as for all subsequent authors, refers to the limiting behavior of the optimal solution as the cost c of observation tends to zero. For discrete time processes, (Bickel and Yahav 1967, 1968, 1969a,b) provided an attractive large sample approximation to the optimal stopping times which they called asymptotically pointwise optimal (APO). A family of stopping times $\{N_c \in \mathcal{N} : c > 0\}$ is APO with respect to the sequence $\{Z_n(c), n \geq 1\}$, if for any other family of stopping times $\{J_c \in \mathcal{N} : c > 0\}$

$$\limsup_{c \to 0} \frac{Z_{N_c}(c)}{Z_{J_c}(c)} \leq 1, \quad a.s.$$

They describe methods for finding a family of stopping times which is APO and they have shown that the APO rule, under some assumptions, is asymptotically optimal (AO), that is, the ratio of the Bayes risk of the APO rule and the Bayes risk of the optimal stopping rule tends to one as the cost per unit sample approaches zero. Later the second-order efficiency of the APO rule is discussed in Woodroofe (1981); Rehaïlia (1984) and Takada (2001).

In this paper, the problem of Bayes sequential estimation of an unknown parameter of a distribution belonging to a particular exponential family is considered. In Sect. 2 a particular exponential family is described. In Sect. 3 the APO and AO families of stopping times are given.

2 A particular exponential family of distributions

Let us consider the one-parameter exponential family of distributions with probability density function

$$f(x, \eta) = \exp[s(x)A(\eta) + B(\eta) + q(x)]\mathbf{1}_D(x) \tag{1}$$

for which

(i) η is an unknown parameter such that $\eta \in \mathcal{H}$,with $\mathcal{H}$ an open set of the real line
(ii) D is a Borel set that does not depend on η, with $\mathbf{1}_D$ the indicator function of the set D
(iii) $A(\eta)$ and $B(\eta)$ are differentiable real functions defined on $\mathcal{H}$
(iv) $A(\eta) < 0$ and $A'(\eta) \neq 0$, for any $\eta \in \mathcal{H}$
(v) for any $\eta \in \mathcal{H}$,

$$2A(\eta)B'(\eta)/A'(\eta) = k, \tag{2}$$

where k is positive (free from η).

From condition (2) we get

$$B(\eta) = \frac{k}{2} \ln |A(\eta)| + k_1, \quad k_1 \in \mathbb{R}.$$

Let $\vartheta = -A(\eta) > 0$, $\Theta = \{\vartheta : \vartheta = -A(\eta), \eta \in \mathcal{H}\}$. Then the family of distributions (1) can be written in the form

$$f(x, \vartheta) = h(x)\vartheta^{k/2} \exp[-s(x)\vartheta],$$

where $h(x) = \exp[q(x)+k_1]\mathbf{1}_D(x)$. The exponential family of distributions described above was introduced in Rahman and Gupta (1993). Table 1 contains some distributions that belong to this family ($\eta > 0$).

The following theorem (see Rahman and Gupta 1993) provides a property of the family of distributions considered and will be needed in the next section.

Theorem 1 *In a one-parameter exponential family of the form given by* (1)*, the statistic $s(X)$ has a gamma $\mathcal{G}(k/2, \vartheta)$ distribution if and only if the condition* (2) *is satisfied.* □

Remark 1 The random variable $2\vartheta s(X)$ has $\mathcal{G}(k/2, 1/2)$ distribution.

By the fact given above, the family of distributions with density function given by (1), satisfying the conditions (i)–(v) with k positive integer, is called transformed chi-square family (see Rahman and Gupta 1993). In the case $k = 2$ the family is called the time-transformed model (see Barlow and Proschan 1988).

The transformed chi-square family of distributions was considered in López-Blázquez et al. (1997) and López-Blázquez (1998) in the context of UMVUE estimation, and in Jozani et al. (2002) in the context of admissible minimax estimation of a bounded parameter ϑ under scale-invariant squared-error loss. In this paper the Bayes sequential estimation of the parameter ϑ under asymmetric LINEX loss function is considered.

3 Bayes sequential estimation of the unknown parameter ϑ

Let $X_1, X_2, \ldots$ be i.i.d. according to a distribution from the family described above. Consider the problem of estimation ϑ sequentially under a loss function of the following form

$$L_n(\vartheta, d) = L(\vartheta, d) + cn,$$

where $L(\vartheta, d)$ is the loss connected with the error of estimation and c is the cost per one observation. It is assumed that $L(\vartheta, d)$ is the LINEX loss of the following form

$$L(\vartheta, d) = b\{\exp[a(d - \vartheta)] - a(d - \vartheta) - 1\}, \tag{3}$$

Table 1 Members of the particular exponential family of distributions

Name of distribution with p.d.f.	$s(x)$	k	ϑ
Normal $\mathcal{N}(\mu, \eta^2)$, $\mu \in \mathbb{R}$—known			
$f(x;\eta) = \frac{1}{\sqrt{2\pi}\eta} \exp\left[-\frac{(x-\mu)^2}{2\eta^2}\right]$	$\frac{1}{2}(x-\mu)^2$	1	$\frac{1}{\eta^2}$
Lognormal $\mathcal{LN}(\mu, \eta^2)$, $\mu \in \mathbb{R}$—known			
$f(x;\eta) = \frac{1}{\sqrt{2\pi}\eta x} \exp\left[-\frac{(\ln(x)-\mu)^2}{2\eta^2}\right] \mathbf{1}_{(0,\infty)}(x)$	$\frac{1}{2}[\ln(x)-\mu]^2$	1	$\frac{1}{\eta^2}$
Gamma $\mathcal{G}(\alpha, \eta)$, $\alpha > 0$—known			
$f(x;\eta) = \frac{x^{\alpha-1}}{\Gamma(\alpha)} \eta^\alpha \exp(-\eta x) \mathbf{1}_{(0,\infty)}(x)$	x	2α	η
Pareto $\mathcal{P}a(x_0, \eta)$, $x_0 > 0$—known			
$f(x;\eta) = \frac{\eta}{x_0}\left(\frac{x_0}{x}\right)^{\eta+1} \mathbf{1}_{(x_0,\infty)}(x)$	$\ln\left(\frac{x}{x_0}\right)$	2	η
Weibull $\mathcal{W}e(\alpha, \eta)$, $\alpha > 0$—known			
$f(x;\eta) = \alpha\eta^\alpha x^{\alpha-1} \exp[-(x\eta)^\alpha] \mathbf{1}_{(0,\infty)}(x)$	x^α	2	η^α
Maxwell $\mathcal{M}a(\eta)$			
$f(x;\eta) = \frac{2}{\sqrt{2\pi}\eta^3} x^2 \exp\left(-\frac{x^2}{2\eta^2}\right) \mathbf{1}_{(0,\infty)}(x)$	$\frac{1}{2}x^2$	6	$\frac{1}{\eta^2}$
Power $\mathcal{P}o(\lambda, \eta)$, $\lambda > 0$—known			
$f(x;\eta) = \frac{\eta x^{\eta-1}}{\lambda^\eta} \mathbf{1}_{(0,\lambda)}(x)$	$\ln\left(\frac{\lambda}{x}\right)$	2	η
Laplace $\mathcal{L}a(\mu, \eta)$, $\mu \in \mathbb{R}$—known			
$f(x;\eta) = \frac{1}{2\eta} \exp\left(-\frac{\lvert x-\mu\rvert}{\eta}\right)$	$\lvert x-\mu\rvert$	2	$\frac{1}{\eta}$
Negative exponential $\mathcal{NE}(\mu, \eta)$, $\mu \in \mathbb{R}$—known			
$f(x;\eta) = \eta \exp[-\eta(x-\mu)] \mathbf{1}_{(\mu,\infty)}(x)$	$x-\mu$	2	η
Inverse Gamma $\mathcal{IG}am(\alpha, \eta)$, $\alpha > 0$—known			
$f(x;\eta) = \frac{\eta^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} \exp\left(-\eta\frac{1}{x}\right) \mathbf{1}_{(0,\infty)}(x)$	$\frac{1}{x}$	2α	η
Inverse Gaussian $\mathcal{IG}(\mu, \eta)$, $\mu \in \mathbb{R}$—known			
$f(x;\eta) = \sqrt{\frac{\eta}{2\pi x^3}} \exp\left[-\frac{\eta(x-\mu)^2}{2\mu^2 x}\right] \mathbf{1}_{(0,\infty)}(x)$	$\frac{(x-\mu)^2}{2\mu^2 x}$	1	η
Extreme values distribution of type I, $\lambda > 0$ – known			
$f(x;\eta) = \lambda \exp(\lambda x) \exp(-\lambda\eta) \exp[-\exp(\lambda x)\exp(-\lambda\eta)]$	$\exp(\lambda x)$	2	$\exp(-\lambda\eta)$
Modified extreme value distribution			
$f(x;\eta) = 1/\eta \exp(x) \exp\left[-(\exp(x-1))/\eta\right] \mathbf{1}_{(0,\infty)}(x)$	$\exp(x-1)$	2	η^{-1}

where $a \neq 0$ is a shape parameter and $b > 0$ is a factor of proportionality. The loss function given by (3) was introduced by Varian (1975). If a is positive (negative), then under(over)-estimation is considered to be more serious than over(under)-estimation of the same magnitude, and vice versa.

Let $\Pi_{\alpha,\lambda}$ be a gamma $\Gamma(\alpha, \lambda)$ prior distribution of the parameter ϑ with p.d.f. of the form

$$\pi_{\alpha,\lambda}(\vartheta) = \frac{\vartheta^{\alpha-1}}{\Gamma(\alpha)} \lambda^\alpha \exp(-\lambda\vartheta), \quad \alpha > 0, \ \lambda > 0.$$

Denote $S_n(X_1, \ldots, X_n) = \sum_{i=1}^n s(X_i)$.

Lemma 1 *If the parameter ϑ has the gamma $\Gamma(\alpha, \lambda)$ prior distribution, then a posteriori distribution, given $X_1, \ldots, X_n$, is the gamma distribution $\Gamma(\alpha_n, \lambda_n)$, where $\alpha_n = \alpha + n\frac{k}{2}$ and $\lambda_n = \lambda + S_n$.*

It is well known fact that the Bayes sequential procedure $\delta^\pi = (N, d^\pi(X_1, \ldots, X_N))$ with respect to a priori distribution π is such that for a given stopping rule N, the estimator d^π is the Bayes estimate.

Lemma 2 *Under the LINEX loss function given by (3) and for $N = n$, the Bayes estimator d^π of the parameter ϑ is of the form*

$$d^\pi(X_1, \ldots, X_n) = -\frac{1}{a} \log\{E^\pi[\exp(-a\vartheta) \mid X_1, \ldots, X_n]\}, \tag{4}$$

and the posterior expected loss is

$$E^\pi[L(\vartheta, d^\pi) \mid X_1, \ldots, X_n] = ab[E^\pi(\vartheta \mid X_1, \ldots, X_n) - d^\pi(X_1, \ldots, X_n)].$$

Proof It is well known that an estimator minimizing the risk $E^\pi[E_\vartheta(L(\vartheta, d)]$ can be obtained by selecting, for every $x \in \mathcal{X}$, the value $d(x)$ which minimizes the posterior expected loss $E^\pi[L(\vartheta, d) \mid X_1, \ldots, X_n]$. Under the LINEX loss function, the posterior expected loss has the following form

$$\begin{aligned} E^\pi[L(\vartheta, d) \mid X_1, \ldots, X_n] = b \exp[-ad]E^\pi[\exp(a\vartheta) \mid X_1, \ldots, X_n] \\ - abE^\pi(\vartheta \mid X_1, \ldots, X_n) + abd - b, \end{aligned}$$

and it attains its minimum for d^π defined by (4). The form of the posterior expected loss for the estimator d^π follows from straightforward calculations. □

From Lemma 2 we have that in our case (under a gamma $\mathcal{G}(\alpha, \beta)$ prior distribution $\Pi_{\alpha,\lambda}$), assuming that $a > -\lambda$, the Bayes estimator of the parameter ϑ is of the form

$$d^{\Pi_{\alpha,\lambda}}(X_1, \ldots, X_n) = \frac{\alpha_n}{a} \log\left(1 + \frac{a}{\lambda_n}\right) = \frac{1}{a}\left(\alpha + n\frac{k}{2}\right) \log\left(1 + \frac{a}{\lambda + S_n}\right).$$

Hence, for $b = 1$ (without loss of generality),

$$E^{\Pi_{\alpha,\lambda}}[L(\vartheta, d^{\Pi_{\alpha,\lambda}}) \mid X_1, \ldots, X_n] = \alpha_n \left[\frac{a}{\lambda_n} - \log\left(1 + \frac{a}{\lambda_n}\right)\right] =: Y_n. \tag{5}$$

Thus finding an optimal procedure $\delta^\pi = (N^*, d^\pi(X_1, \ldots, X_{N^*}))$ for the Bayes sequential problem reduces to finding an optimal stopping rule for the sequence

$$Z_n(c) = Y_n + cn, \quad n = 1, 2, \ldots$$

In our case

$$Z_n(c) = \left(n\frac{k}{2} + \alpha\right)\left[\frac{a}{S_n + \lambda} - \log\left(1 + \frac{a}{S_n + \lambda}\right)\right] + cn. \tag{6}$$

Finding the Bayes stopping rule N^* seems extremely difficult. We will derive the APO and AO families of stopping times with respect to the sequence $\{Z_n(c), n \geq 1\}$ given by (6). To obtain our results we use the following theorem (see Bickel and Yahav 1968) giving the sufficient conditions for a family of stopping times to be APO with respect to the sequence $\{Z_n(c), n \geq 1\}$.

Theorem 2 *If*

(i) $P(Y_n > 0) = 1$ *for any* $n \geq 1$,
(ii) $n^{\beta} Y_n \to V$ *a.s. as* $n \to \infty$, *for some* $\beta > 0$, *where* $P(0 < V < \infty) = 1$,

then the family $\{N_c : c > 0\}$ *of stopping times*

$$N_c = \inf\{n \geq 1 : Y_n \leq cn/\beta\}$$

is APO with respect to the sequence $\{Z_n(c), n \geq 1\}$. □

Applying Theorem 2 to the sequence (5), and assuming that $a > -\lambda$, we have the following theorem.

Theorem 3 *In the problem of sequentially estimating the parameter* ϑ *of a distribution belonging to the underlying exponential family, the family of stopping times* $\{N_c : c > 0\}$ *defined by*

$$N_c = \inf\left\{n \geq 1 : \left(\frac{nk}{2} + \alpha\right)\left[\frac{a}{S_n + \lambda} - \log\left(1 + \frac{a}{S_n + \lambda}\right)\right] \leq cn\right\} \tag{7}$$

is APO with respect to the sequence $\{Z_n(c), n \geq 1\}$ *given by* (6).

Proof It is sufficient to show, that the stochastic process $\{Y_n, n \geq 1\}$, given by (5), fulfils the assumptions of Theorem 2. It is obvious that the process $\{Y_n, n \geq 1\}$ is $\mathcal{F}_n$ measurable, where the σ-fields $\mathcal{F}_n \subset \mathcal{F}$ and $\mathcal{F}_n$ are increasing in n. It is easy to see that $P(Y_n > 0) = 1$ for any $n \geq 1$. The process nY_n can be written in the form

$$nY_n = n^2\left(\frac{\alpha}{n} + \frac{k}{2}\right)\left[\frac{a}{S_n + \lambda} - \log\left(1 + \frac{a}{S_n + \lambda}\right)\right].$$

It is easy to see that

$$\frac{a^2}{2(S_n + \lambda)^2} - \frac{a^3}{3(S_n + \lambda)^3} \leq \frac{a}{S_n + \lambda} - \log\left(1 + \frac{a}{S_n + \lambda}\right) \leq \frac{a^2}{2(S_n + \lambda)^2}$$

for $a > 0$, and

$$\frac{a^2}{2(S_n+\lambda)^2} \leq \frac{a}{S_n+\lambda} - \log\left(1+\frac{a}{S_n+\lambda}\right) \leq \frac{a^2}{2(S_n+\lambda)^2} - \frac{a^3}{3(S_n+\lambda)^3(1+a/\lambda)^3}$$

for $a < 0$, from which we get

$$nY_n \to V = \frac{k}{2}\frac{a^2}{2[E(s(X))]^2} = \frac{a^2\vartheta^2}{k} \quad a.s. \text{ as } n \to \infty,$$

by using the fact that $E(s(X)) = \frac{k}{2\vartheta}$ (see Theorem 1). Thus from Theorem 2 the family of stopping times given by (7) is APO. □

Moreover Bickel and Yahav (1968) showed that in many point estimation and hypothesis testing problems, the APO rules are AO in the sense of Kiefer and Sacks (1963).

Following Kiefer and Sacks (1963) we say that a family of stopping times $\{N_c \in \mathcal{N} : c > 0\}$ is AO with respect to the sequence $\{Z_n(c), n \geq 1\}$, if for any other family of stopping times $\{J_c \in \mathcal{N} : c > 0\}$

$$\limsup_{c\to 0} \frac{E(Z_{N_c}(c))}{E(Z_{J_c}(c))} \leq 1.$$

The following theorem is a slight modification of Theorem 3.1 at the paper Bickel and Yahav (1968), and it gives sufficient conditions for a family of stopping times to be AO with respect to the sequence $\{Z_n(c), n \geq 1\}$.

Theorem 4 *If the sequence $\{Y_n, n \geq 1\}$ obeys the assumptions of Theorem 2 and*

$$\sup_{n\geq n_0} n^\beta E(Y_n) < \infty,$$

where n_0 is such that $E(Y_n) < \infty$ for $n \geq n_0$, then the family $\{N_c, c \geq 0\}$ of stopping times

$$N_c = \inf\{n \geq n_0 : Y_n \leq cn/\beta\}, \quad c > 0,$$

is AO with respect to the sequence $\{Z_n(c), n \geq 1\}$. □

Using Theorem 4 we obtain the following result.

Theorem 5 *In the problem of sequentially estimating the parameter ϑ of a distribution belonging to the underlying exponential family, the family $\{N_c : c > 0\}$ of stopping times*

$$N_c = \inf\left\{n \geq [4/k]+1 : \left(\frac{nk}{2}+\alpha\right)\left[\frac{a}{S_n+\lambda} - \log\left(1+\frac{a}{S_n+\lambda}\right)\right] \leq cn\right\} \quad (8)$$

is AO with respect to the sequence $\{Z_n(c), n \geq 1\}$ given by (6).

Proof It is sufficient to show, that $\sup_{n\geq n_0} nE(Y_n) < \infty$. Let us notice that $E(Y_n) < \infty$ for $n > 4/k$, and $\lim_{n\to\infty} nE(Y_n) = a^2\vartheta^2/k$. Thus, the assumptions of Theorem 4 are satisfied with $\beta = 1$, and the family of stopping times given by (8) is AO. □

Example As an example, assume that a life test on machines can be conducted sequentially in time beginning at $t = 0$. Let X_i, $i = 1, 2, \ldots$, represent the time at which the machine i would fail if it were allowed to operate indefinitely. Let us also assume that the times $X_1, X_2, \ldots$ are the random variables from the Weibull distribution $\mathcal{W}(2, \eta)$ given by

$$f(x;\eta) = 2x\eta^2 \exp\left(-x^2\eta^2\right) \mathbf{1}_{(0\infty)}(x),$$

and that our prior knowledge about the parameter $\vartheta = \eta^2$ is it has the gamma distribution $\mathcal{G}(\alpha, \beta)$. We want to estimate the parameter ϑ under the loss function given by (3). In this case the process $Z_n(c)$ is given by (6) with $S_n = \sum_{i=1}^n X_i^2$. From Theorems 3 and 5 we have that the family of stopping times $\{N_c : c > 0\}$ defined by

$$N_c = \inf\left\{n \geq 3 : (\alpha + n)\left[a/\left(\lambda + \sum_{i=1}^n X_i^2\right) - \log\left(1 + a/\left(\lambda + \sum_{i=1}^n X_i^2\right)\right)\right] \leq cn\right\}$$

is APO and AO. From Lemma 2, the Bayes estimator of the parameter $\vartheta = \eta^2$ takes the form

$$d^{\Pi_{\alpha,\lambda}}\left(X_1, \ldots, X_{N_c}\right) = \frac{1}{a}(\alpha + N_c)\log\left[1 + a/\left(\lambda + \sum_{i=1}^{N_c} X_i^2\right)\right].$$

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