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On estimating the mean of the selected normal population under the LINEX loss function

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Abstract. Following Parsian and Farsipour (1999), we consider the problem of estimating the mean of the selected normal population, from two normal populations with unknown means and common known variance, under the LINEX loss function. Some admissibility results for a subclass of equivariant estimators are derived and a sufficient condition for the inadmissibility of an arbitrary equivariant estimator is provided. As a consequence, several of the estimators proposed by Parsian and Farsipour (1999) are shown to be inadmissible and better estimators are obtained.

Key words: admissible estimators, equivariant estimators, inadmissible estimators, LINEX loss function, mean of the selected population, natural selection rule

1 Introduction

The problems of estimation after selection have been studied by Sarkadi (1967), Putter and Rubinstein (1968), Dahiya (1974), Hseih (1981), Cohen and Sackrowitz (1982), Sackrowitz and Samuel-Cahn (1984), Venter (1988), Vellaisamy (1992), Misra (1994) and Parsian and Farsipour (1999). Some work has also been done for optimizing both, selection and estimation, in the decision-theoretic setup, for which the reader may refer to Gupta and Miescke (1990, 1993), Cohen and Sackrowitz (1988) and other references cited therein.

Dahiya (1974) considered estimation after selection from two normal populations, having a common known variance, and proposed six different estimators for the mean of the selected population (one associated with the larger sample mean) and studied the performances of these estimators, numerically, under the squared loss function.

Note that any symmetric loss function, such as squared error loss function, assigns the same penalties to overestimation and underestimation. In some situations, overestimation may be considered more serious than the underestimation or vice versa. To deal with such situations, Varian (1975) proposed the following alternative loss function

$$L(\theta,\delta) = e^{a(\delta-\theta)} - a(\delta-\theta) - 1, \quad a \neq 0.$$

When $a > 0$, the loss function increases almost exponentially for positive $(\delta - \theta)$ and almost linearly otherwise, so overestimation is more heavily penalized than underestimation. When $a < 0$, the linear-exponential rises are interchanged and underestimation is considered more costly than overestimation. In the literature, such loss functions are called LINEX (linear-exponential) loss functions. Some of the researchers who have considered LINEX loss function for estimation are: Zellner (1986), Kuo and Dey (1990), Sadooghi-Alvandi (1990), Basu and Ebrahimi (1991), Sadooghi-Alvandi and Parsian (1992) and Madi (1997).

Following Dahiya (1974), Parsian and Farsipour (1999) studied estimation after selection from two normal populations, having a common known variance, under the criterions of bias and LINEX loss function. They proposed seven different estimators for the mean of the selected population and obtained expressions for the biases and risk functions of these estimators. Using these expressions, they studied performances of various estimators, numerically, under the criterions of bias and LINEX loss function.

In this paper, we continue the study of Parsian and Farsipour (1999) by deriving some decision-theoretic results for the problem under the LINEX loss function. In particular, we show that several of the estimators proposed by Parsian and Farsipour (1999) are inadmissible under the LINEX loss function and we obtain better estimators. Here we note that Parsian and Farsipour (1999) considered not only the criterion of expected LINEX loss but also the criterion of bias.

Let X_1 and X_2 be two independent random variables representing the populations Π_1 and Π_2, respectively, which are normally distributed with respective unknown means θ_1 and θ_2 and have a common known variance σ^2. Throughout, the following notations will be adopted: $\underline{X} = (X_1, X_2)$, $\underline{\theta} = (\theta_1, \theta_2)$, $Y_1 = \min(X_1, X_2)$, $Y_2 = \max(X_1, X_2)$, $\mu_1 = \min(\theta_1, \theta_2)$, $\mu_2 = \max(\theta_1, \theta_2)$, $\underline{Y} = (Y_1, Y_2)$, $Y = Y_1 - Y_2$, $\mu = \mu_2 - \mu_1$, so that $Y \leq 0$, *w.p.* 1 and $\mu \geq 0$. Also, throughout, $\Phi(.)$ and $\phi(.)$ will denote, respectively, the cumulative distribution function (cdf) and the probability density function (pdf) of the $N(0,1)$ distribution, $\Re$ will denote the real line, $\Re_+$ will denote the non-negative part of the real line, $\Re^2$ will denote the two dimensional Euclidean space, for a fixed $\underline{\theta} \in \Re^2$, $P_{\underline{\theta}}(.)$ will denote the probability measure induced by $\underline{X}$ and $E_{\underline{\theta}}(.)$ will denote the expectation under $P_{\underline{\theta}}$.

For the goal of selecting the unknown population associated with the larger mean μ_2, consider the natural selection rule δ^N, according to which the population corresponding to the larger observation Y_2 is selected. Optimum properties of the natural selection rule δ^N have been established by Eaton (1967). Let M denote the index of the selected population. We desire to estimate

$$\theta_M = \begin{cases} \theta_1, & \text{if } X_1 > X_2 \\ \theta_2, & \text{if } X_1 < X_2, \end{cases} \tag{1.1}$$

under the LINEX loss function

$$L(\underline{\theta},\delta) = e^{a(\delta-\theta_M)} - a(\delta - \theta_M) - 1, \tag{1.2}$$

where $a \neq 0$ is a given constant. Note that θ_M, given by (1.1), is a random parameter. The given estimation problem is invariant under the location group of transformations and under the group of permutations. Therefore, it is natural to consider only those estimators δ which are permutation and location invariant, i.e. estimators satisfying $\delta(X_1, X_2) = \delta(X_2, X_1)$ and $\delta(X_1 + c, X_2 + c) = \delta(X_1, X_2) + c$, $\forall c \in \Re$. Any such estimator will be of the form

$$\delta_\psi(\underline{Y}) = Y_2 + \psi(Y), \tag{1.3}$$

for some real valued function $\psi(.)$, defined on the non-positive part of the real line. An estimator of the form (1.3) is called an equivariant estimator. Let $\mathscr{D}_1$ denote the class of all equivariant estimators. For an equivariant estimator $\delta \in \mathscr{D}_1$, the risk function $R(\underline{\theta},\delta) = E_{\underline{\theta}}[L(\underline{\theta},\delta(\underline{Y}))]$ depends on $\underline{\theta}$ only through $\mu = \mu_2 - \mu_1$ (cf. Ferguson (1967), pp. 149). Therefore, for notational convenience, we denote $R(\underline{\theta},\delta)$ by $R_\mu(\delta)$.

In Section 2 of this paper, we derive admissible (or inadmissible) estimators within a subclass of $\mathscr{D}_1$, under the LINEX loss function, given by (1.1). In Section 3, we provide a sufficient condition for the inadmissibility of equivariant estimators under the LINEX loss function. Section 4 deals with some applications of results obtained in Section 3.

2 Some admissibility results

Consider the following subclass of equivariant estimators

$$\mathscr{D}_2 = \{\delta_c(.) : \delta_c(\underline{Y}) = Y_2 + c, c \in \Re\}. \tag{2.1}$$

Among the various estimators proposed by Parsian and Farsipour (1999), two estimators, $\delta_0(\underline{Y}) = Y_2$ and $\delta_{c_0}(\underline{Y}) = Y_2 - a\sigma^2/2$ belong to the class $\mathscr{D}_2$. The estimator δ_0 (δ_{c_0}) is the unique generalized Bayes estimator of θ_M under the squared error (LINEX) loss function for the non-informative prior, i.e. the Lebesgue measure on $\Re^2$, (cf. Dahiya (1974) and Parsian and Farsipour (1999)). We will compare the performances of various estimators in the class $\mathscr{D}_2$, under the LINEX loss function, by deriving all admissible (or inadmissible) estimators within the class $\mathscr{D}_2$. The following lemma will be useful in doing so.

Lemma 2.1: *Let $S = Y_2 - \theta_M$ and, for a given real constant d, define the functions $h_d(\mu) = \Phi(d + \mu) + \Phi(d - \mu)$, $\mu \geq 0$ and $s(x) = x^2 + \ln 2 + \ln(\Phi(x))$, $x \in \Re$. Also, let x_0 $(=-1.2395\ldots) < 0$ be the root of the equation $s(x) = 0$. Then,*

(i) *the pdf of S is given by*

$$f_S(s|\mu) = \left[\Phi\left(\frac{s+\mu}{\sigma}\right) + \Phi\left(\frac{s-\mu}{\sigma}\right)\right]\frac{1}{\sigma}\phi\left(\frac{s}{\sigma}\right), \quad s \in \Re,$$

(ii) *for* $d < 0$ (>0), *the function* $h_d(\mu)$ *is an increasing (a decreasing) function of* $\mu \in \Re_+$,

(iii) $s(x) < 0$, $\forall x \in (x_0, 0)$ *and* $s(x) > 0$, $\forall x \in (-\infty, x_0) \cup \Re_+$.

Proof. (i) The cdf of S is given by

$$F_S(s|\mu) = P_{\underline{\theta}}(X_1 < X_2, X_2 - \theta_2 \le s) + P_{\underline{\theta}}(X_1 > X_2, X_1 - \theta_1 \le s)$$

$$= \int_{-\infty}^{s/\sigma} \Phi\left(z + \frac{\mu}{\sigma}\right)\phi(z)\,dz + \int_{-\infty}^{s/\sigma} \Phi\left(z - \frac{\mu}{\sigma}\right)\phi(z)\,dz.$$

Now the assertion follows on differentiating both sides with respect to s.

(ii) Follows through differentiation.

(iii) For $x \in \Re$, we have $s'(x) = t(x)/\Phi(x)$, where $t(x) = 2x\Phi(x) + \phi(x)$. Now, for $x \in \Re$, $t'(x) = 2\Phi(x) + x\phi(x)$ and $t''(x) = (3 - x^2)\phi(x)$. Since, $\lim_{x \downarrow -\infty} t'(x) = 0$, $t'(0) = 1$ and $\lim_{x \uparrow \infty} t'(x) = 2$, it follows that $t'(x) < 0$ for $x < y_0$ and $t'(x) > 0$ for $x > y_0$, where $y_0 \in (-\sqrt{3}, 0)$ is the solution of $t'(y) = 0$. Also, since, $\lim_{x \downarrow -\infty} t(x) = 0$, $t(0) = \frac{1}{\sqrt{2\pi}}$ and $\lim_{x \uparrow \infty} t(x) = \infty$, it follows that $t(x) < 0$ for $x < z_0$ and $t(x) > 0$ for $x > z_0$, where $z_0 \in (y_0, 0)$ is the solution of the equation $t(z) = 0$. Thus, it follows that $s(x)$ is a decreasing function of x if $x < z_0$ and it is an increasing function of x if $x > z_0$. Now the result follows on noting that $\lim_{x \downarrow -\infty} s(x) = \infty$ and $s(0) = 0$.

Theorem 2.1: (i) *Let* $b_0 = -\frac{1}{a}\left[\frac{a^2\sigma^2}{2} + \ln 2 + \ln\left\{\Phi\left(\frac{a\sigma}{\sqrt{2}}\right)\right\}\right]$ *and* $c_0 = -\frac{a\sigma^2}{2}$. *Then, under the LINEX loss function (1.2), the estimators* $\delta_c(.)$, *for* $b_0 \le c \le c_0$, *are admissible within the class* $\mathcal{D}_2$.

(ii) *For each* $\mu \in \Re_+$, *the risk function* $R_\mu(\delta_c)$ *is a decreasing function of* c *if* $c < b_0$ *and it is an increasing function of* c *if* $c > c_0$. *In particular, under the LINEX loss function (1.2), the estimators* $\delta_c(.)$, *for* $c \in (-\infty, b_0) \cup (c_0, \infty)$, *are inadmissible even within the class* $\mathcal{D}_2$.

Proof. Define, $K(\mu) = -\ln[E_{\underline{\theta}}(e^{aS})]/a$, $\mu \in \Re_+$, where $S = Y_2 - M$. Then, for fixed $\mu \in \Re_+$, the risk function

$$R_\mu(\delta_c) = e^{ac}E_{\underline{\theta}}(e^{aS}) - ac - aE_{\underline{\theta}}(S) - 1 \tag{2.2}$$

is minimized at $c = K(\mu)$. On using Lemma 2.1 (i), we have

$$K(\mu) = -\frac{a\sigma^2}{2} - \frac{1}{a}\ln\left[\Phi\left\{\frac{1}{\sqrt{2}}\left(a\sigma + \frac{\mu}{\sigma}\right)\right\} + \Phi\left\{\frac{1}{\sqrt{2}}\left(a\sigma - \frac{\mu}{\sigma}\right)\right\}\right].$$

Now, on using Lemma 2.1 (ii), it follows that

$$\inf_{\mu \in \Re_+} K(\mu) = K(0) = b_0, \quad \text{and} \quad \sup_{\mu \in \Re_+} K(\mu) = \lim_{\mu \uparrow \infty} K(\mu) = c_0. \tag{2.3}$$

(i) Since $K(\mu)$ is a continuous function of μ, from (2.3), it follows that $K(\mu)$ assumes all values in the interval $[b_0, c_0)$. Thus, we conclude that each $c \in [b_0, c_0)$ minimizes the risk function $R_\mu(\delta_c)$, given by (2.2), at some $\mu \in \Re_+$.

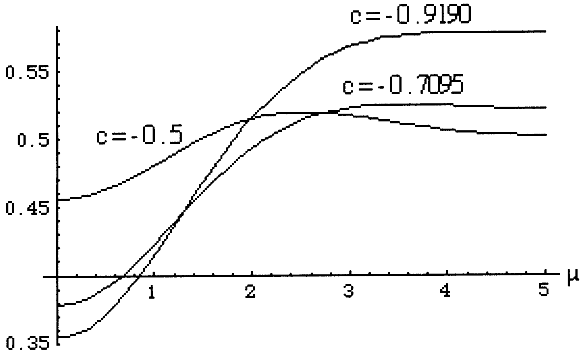

Fig. 1. Plots of $R_\mu(\delta_c)$, for $a = \sigma = 1$ ($c = -0.9190$ $(= b_0)$, -0.7095, -0.5 $(= c_0)$)

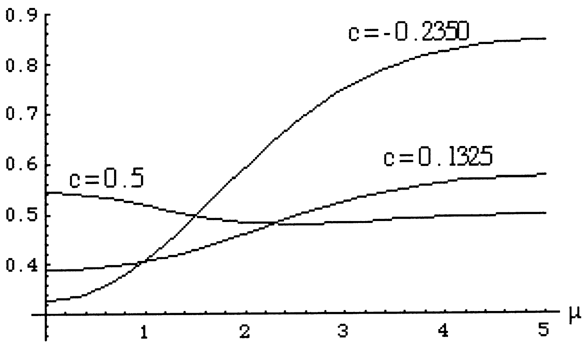

Fig. 2. Plots of $R_\mu(\delta_c)$, for $a = -1$ and $\sigma = 1$ ($c = -0.2350$ $(= b_0)$, 0.1325, 0.5 $(= c_0)$)

This establishes that the estimators δ_c, for $c \in [b_0, c_0)$, are admissible within the class $\mathscr{D}_2$. The admissibility of the estimator δ_{c_0}, within the class $\mathscr{D}_2$, follows from the continuity of the risk function.

(ii) For each fixed $\mu \in \Re_+$, $R_\mu(\delta_c)$ is an increasing function of c, if $c > K(\mu)$, and it is a decreasing function of c, if $c < K(\mu)$. Since, $b_0 \leq K(\mu) \leq c_0$, $\forall \mu \in \Re_+$, the result follows.

For $a = \sigma = 1$ and $c = -0.9190\ (= b_0)$, -0.7095, $-0.5\ (= c_0)$, plots of functions $R_\mu(\delta_c)$, as functions of μ, are displayed in Figure 1. Similarly, for $a = -1$, $\sigma = 1$ and $c = -0.2350\ (= b_0)$, 0.1325, $0.5\ (= c_0)$, plots of functions $R_\mu(\delta_c)$, as functions of μ, are displayed in Figure 2.

Remark: (i) *It follows from Theorem 2.1 (i) that the generalized Bayes estimator $\delta_{c_0}(\underline{Y})$ is admissible within the class $\mathscr{D}_2$. For $a = \sigma = 1$, it is evident from Figure 1 that the estimator δ_c, with $c = -0.7095$, is significantly better than the generalized estimator δ_{c_0} for small values of μ and, for large values of μ, there is not much difference in their performances. This suggests that, for $a = \sigma = 1$, the estimator δ_c, with $c = -0.7095$, should be preferred over the generalized Bayes estimator δ_{c_0}. Similarly, for $a = -1$ and $\sigma = 1$, Figure 2 suggests that the estimator δ_c, with $c = 0.1325$, should be preferred over the generalized Bayes estimator δ_{c_0}.*

(iv) *On using Theorem 2.1 (ii) along with Lemma 2.1 (iii), it follows that the estimator δ_0 is inadmissible even within the class $\mathscr{D}_2$, provided* $a \in (-\infty, \sqrt{2}x_0/\sigma) \cup (0, \infty)$.

3 A sufficient condition for inadmissibility

Now consider the class $\mathscr{D}_1$ of all equivariant estimators. In this section, we will exploit the orbit by orbit improvement technique of Brewster and Zidek (1974) to derive a sufficient condition for the inadmissibility of equivariant estimators. The following lemma will be useful in deriving the sufficient condition.

Lemma 3.1: (i) *For $y \le 0$, the conditional pdf of $S = Y_2 - \theta_M$ given that $Y = y$ $(Y = Y_1 - Y_2)$ is given by*

$$f_{S\,|\,Y=y}(x|\mu) = \frac{\sqrt{2}}{\sigma}\left[\frac{\phi\left(\frac{y+\mu}{\sqrt{2}\sigma}\right)\phi\left\{\frac{\sqrt{2}}{\sigma}\left(x+\frac{y+\mu}{2}\right)\right\}+\phi\left(\frac{y-\mu}{\sqrt{2}\sigma}\right)\phi\left\{\frac{\sqrt{2}}{\sigma}\left(x+\frac{y-\mu}{2}\right)\right\}}{\phi\left(\frac{y+\mu}{\sqrt{2}\sigma}\right)+\phi\left(\frac{y-\mu}{\sqrt{2}\sigma}\right)}\right], \quad x \in \Re.$$

(ii) *For $y \le 0$,*

$$E_{\underline{\theta}}(e^{aS} \mid Y = y) = e^{-ay/2}e^{a^2\sigma^2/4}\frac{e^{-a\mu/2} + e^{\mu(y/\sigma^2+a/2)}}{1 + e^{\mu y/\sigma^2}}.$$

(iii) *For $\alpha > 0$ and $\beta \in \Re$, define*

$$\xi_{\alpha,\beta}(\mu) = \frac{e^{-\beta\mu} + e^{-(2\alpha-\beta)\mu}}{1 + e^{-2\alpha\mu}}, \quad \mu \in \Re_+. \tag{3.1}$$

Then, for $\beta \in (-\infty, 0) \cup (2\alpha, \infty)$, $\xi_{\alpha,\beta}(\mu)$ is an increasing function of $\mu \in \Re_+$, and, for $\beta \in (0, 2\alpha)$, $\xi_{\alpha,\beta}(\mu)$ is a decreasing function of $\mu \in \Re_+$.

(iv) *For fixed $y \le 0$, define $\psi_y(\mu) = -\ln[E_{\underline{\theta}}(e^{aS} \mid Y = y)]/a$, $\mu \in \Re_+$. Then, for $a < 0$,*

$$\inf_{\mu\in\Re_+} \psi_y(\mu) = \frac{y}{2} - \frac{a\sigma^2}{4} = \psi^*(y), \quad \textit{say and} \quad \sup_{\mu\in\Re_+} \psi_y(\mu) = \infty, \tag{3.2}$$

and for $a > 0$

$$\inf_{\mu\in\Re_+} \psi_y(\mu) = \begin{cases} \frac{y}{2} - \frac{a\sigma^2}{4}, & \textit{if } y < -\frac{a\sigma^2}{2} \\ -\infty, & \textit{if } y > -\frac{a\sigma^2}{2} \end{cases} = \psi_{P,I}(y), \ \textit{say}, \tag{3.3}$$

$$\sup_{\mu\in\Re_+} \psi_y(\mu) = \begin{cases} \infty, & \textit{if } y < -\frac{a\sigma^2}{2} \\ \frac{y}{2} - \frac{a\sigma^2}{4}, & \textit{if } y > -\frac{a\sigma^2}{2} \end{cases} = \psi_{P,S}(y), \ \textit{say}. \tag{3.4}$$

Proof. (i) Let $f_Y(y|\mu)$ denote the pdf of Y. Then, the conditional cdf of S given that $Y = y$ $(y \le 0)$ is given by

$$F_{S\,|\,Y=y}(x|\mu) = P_{\underline{\theta}}(S \le x \,|\, Y = y) = \frac{1}{f_Y(y|\mu)} \lim_{h \downarrow 0} \frac{N(h)}{h},$$

where

$$\begin{aligned} N(h) &= P_{\underline{\theta}}(S \le x, y - h < Y \le y) \\ &= \int_{-\infty}^{x/\sigma} \left[\Phi\left(\frac{y+\mu}{\sigma} + z\right) - \Phi\left(\frac{y-h+\mu}{\sigma} + z\right) \right] \phi(z)\, dz \\ &\quad + \int_{-\infty}^{x/\sigma} \left[\Phi\left(\frac{y-\mu}{\sigma} + z\right) - \Phi\left(\frac{y-h-\mu}{\sigma} + z\right) \right] \phi(z)\, dz \end{aligned}$$

$$\Rightarrow \lim_{h \downarrow 0} \frac{N(h)}{h} = \frac{1}{\sigma} \left[\int_{-\infty}^{x/\sigma} \phi\left(\frac{y+\mu}{\sigma} + z\right) \phi(z)\, dz + \int_{-\infty}^{x/\sigma} \phi\left(\frac{y-\mu}{\sigma} + z\right) \phi(z)\, dz \right].$$

Now the result follows on differentiation and using the fact that the pdf of Y is given by

$$f_Y(y|\mu) = \frac{1}{\sigma\sqrt{2}} \left[\phi\left(\frac{y+\mu}{\sigma\sqrt{2}}\right) + \phi\left(\frac{y-\mu}{\sigma\sqrt{2}}\right) \right], \quad y \le 0.$$

(ii) The assertion follows from (i), on using the fact that the moment generating function of a $V \sim N(\vartheta, \rho^2)$ random variable is given by $M_V(t) = \exp(\vartheta t + \rho^2 t^2/2)$.

(iii) Follows through differentiation.

(iv) On using (ii), for $y \le 0$, we get $\psi_y(\mu) = \frac{y}{2} - \frac{a\sigma^2}{4} - \ln[\xi_{\alpha,\beta}(\mu)]/a$, where $\alpha = -\frac{y}{2\sigma^2}$, $\beta = a/2$ and $\xi_{\alpha,\beta}(\mu)$ is given by (3.1). Now, on using (iii), it follows that, for $a \in (-\infty, 0) \cup (0, -2y/\sigma^2)$, $\psi_y(\mu)$ is an increasing function of $\mu \in \Re_+$. Therefore, for $a < 0$ or $y < -a\sigma^2/2 < 0$,

$$\inf_{\mu \in \Re_+} \psi_y(\mu) = \psi_y(0) = \frac{y}{2} - \frac{a\sigma^2}{4} \quad \text{and} \quad \sup_{\mu \in \Re_+} \psi_y(\mu) = \lim_{\mu \uparrow \infty} \psi_y(\mu) = \infty.$$

On using (iii), we also conclude that, for $a \in (-2y/\sigma^2, \infty)$, $\psi_y(\mu)$ is a decreasing function of $\mu \in \Re_+$. Therefore, for $-a\sigma^2/2 < y < 0$ and $a > 0$

$$\inf_{\mu \in \Re_+} \psi_y(\mu) = \lim_{\mu \uparrow \infty} \psi_y(\mu) = -\infty \quad \text{and} \quad \sup_{\mu \in \Re_+} \psi_y(\mu) = \psi_y(0) = \frac{y}{2} - \frac{a\sigma^2}{4}.$$

Hence the assertion follows.

Let $\psi^*(.)$, $\psi_{P,I}(.)$ and $\psi_{P,S}(.)$ be as defined by (3.2), (3.3) and (3.4), respectively. For a real valued function $\psi(.)$, defined on the non-positive part of the real line, let

$$\psi_1(y) = \max[\psi(y), \psi^*(y)], \quad y \le 0, \tag{3.5}$$

and

$$\psi_2(y) = \begin{cases} \psi_{P,I}(y), & \text{if } \psi(y) < \psi_{P,I}(y) \\ \psi(y), & \text{if } \psi_{P,I}(y) \le \psi(y) < \psi_{P,S}(y);\ y \le 0, \\ \psi_{P,S}(y), & \text{if } \psi(y) \ge \psi_{P,S}(y) \end{cases} \tag{3.6}$$

denote the curtailed (truncated) versions of $\psi(.)$.

Theorem 3.1: *For the problem of estimating θ_M under the LINEX loss function (1.2), consider an equivariant estimator $\delta_\psi(\underline{Y}) = Y_2 + \psi(Y)$, where $\psi(.)$ is a real valued function defined on the non-positive part of the real line.*

(i) *Suppose that $a < 0$ and $P_{\underline{\theta}}[\psi(Y) < \psi^*(Y)] > 0$, $\forall \underline{\theta} \in \Re^2$. Then, the estimator $\delta_\psi(\underline{Y})$ is inadmissible and is dominated by $\delta_{\psi_1}(\underline{Y}) = Y_2 + \psi_1(Y)$, where $\psi_1(.)$ is given by (3.5).*

(ii) *Suppose that $a > 0$ and $P_{\underline{\theta}}[\{\psi(Y) < \psi_{P,I}(Y)\} \cup \{\psi(Y) > \psi_{P,S}(Y)\}] > 0$, $\forall \underline{\theta} \in \Re^2$. Then, the estimator $\delta_\psi(\underline{Y})$ is inadmissible and is dominated by $\delta_{\psi_2}(\underline{Y}) = Y_2 + \psi_2(Y)$, where $\psi_2(.)$ is given by (3.6).*

Proof. (i) Suppose that $a < 0$. For $\underline{\theta} \in \Re^2$, consider the risk difference

$$\begin{aligned} \Delta_1(\mu) &= R_\mu(\delta_\psi) - R_\mu(\delta_{\psi_1}) \\ &= E_{\underline{\theta}}[D_{\underline{\theta}}(Y)], \end{aligned}$$

where, for $y \le 0$,

$$D_{\underline{\theta}}(y) = [e^{a\psi(y)} - e^{a\psi_1(y)}]E_{\underline{\theta}}(e^{aS} \mid Y = y) - a[\psi(y) - \psi_1(y)].$$

Fix $y \le 0$. Clearly, if $\psi(y) \ge \psi^*(y)$, then $D_{\underline{\theta}}(y) = 0$, $\forall \underline{\theta} \in \Re^2$. Now suppose that $\psi(y) < \psi^*(y)$. Then, on using Lemma 3.1 (iv) along with the fact that $e^x > 1 + x$, $\forall x \ne 0$, it follows that

$$\begin{aligned} D_{\underline{\theta}}(y) &\ge e^{a[\psi(y) - \psi^*(y)]} - 1 - a[\psi(y) - \psi^*(y)], \quad \forall \underline{\theta} \in \Re^2 \\ &> 0, \quad \forall \underline{\theta} \in \Re^2, \end{aligned}$$

Now, since $P_{\underline{\theta}}[\psi(Y) < \psi^*(Y)] > 0$, $\forall \underline{\theta} \in \Re^2$, it follows that $\Delta_1(\mu) > 0$, $\forall \underline{\theta} \in \Re^2$.

(ii) Similar to the proof of (i) and therefore omitted.

4 Applications

Parsian and Farsipour (1999) proposed, among others, the following estimators

$$\delta_\tau(\underline{Y}) = Y_2, \tag{4.1}$$

$$\delta_{\xi}(\underline{Y}) = Y_2 + \frac{1}{a}\ln\left[1 + (e^{aY} - 1)\Phi\left(\frac{Y}{\sigma\sqrt{2}}\right)\right], \tag{4.2}$$

$$\delta_{3,c}(\underline{Y}) = \begin{cases} \frac{Y_1+Y_2}{2}, & \text{if } Y > -\sqrt{2}c\sigma \\ Y_2, & \text{if } Y \leq -\sqrt{2}c\sigma \end{cases};\ c > 0. \tag{4.3}$$

and

$$\delta_{\eta}(\underline{Y}) = Y_2 - \frac{a\sigma^2}{2}. \tag{4.4}$$

Note that $\delta_\tau(.)$ and $\delta_\eta(.)$ coincide with $\delta_0(.)$ and $\delta_{c_0}(.)$, respectively. Under the LINEX loss function, given by (1.2), simple applications of Theorem 3.1 yield the inadmissibility of estimators given by (4.1)–(4.3). Below, we provide dominating estimators for each of the estimators given by (4.1)–(4.3).

Estimator	**Dominating Estimator**
$\delta_\tau(\underline{Y})$	$\delta_0^*(\underline{Y}) = \begin{cases} Y_2, & \text{if } Y \leq -\frac{\lvert a\rvert}{2}\sigma^2 \\ \frac{Y_1+Y_2}{2} - \frac{a\sigma^2}{4}, & \text{if } Y > -\frac{\lvert a\rvert}{2}\sigma^2 \end{cases}$
$\delta_\xi(\underline{Y})$	$\delta_{\xi,a}^*(\underline{Y}) = \begin{cases} \max[\delta_\xi(\underline{Y}), Y_2 + \psi^*(\underline{Y})], & \text{if } a < 0 \\ \min[\delta_\xi(\underline{Y}), Y_2 + \psi_{P,S}(\underline{Y})], & \text{if } a > 0 \end{cases}$
$\delta_{3,c}(\underline{Y})$,	$\delta_{3,c,a}^*(\underline{Y})$,

where, $\psi^*(.)$ and $\psi_{P,S}(.)$ are given by (3.2) and (3.4), respectively, and, for $a < 0$,

$$\delta_{3,c,a}^*(\underline{Y}) = \begin{cases} \frac{Y_1+Y_2}{2} - \frac{a\sigma^2}{4}, & \text{if } Y \geq \min\left(-\sqrt{2}c\sigma, \frac{a\sigma^2}{2}\right) \\ Y_2, & \text{if } Y < \min\left(-\sqrt{2}c\sigma, \frac{a\sigma^2}{2}\right), \end{cases}$$

for $0 < a < 2\sqrt{2}\frac{c}{\sigma}$,

$$\delta_{3,c,a}^*(\underline{Y}) = \begin{cases} \frac{Y_1+Y_2}{2} - \frac{a\sigma^2}{4}, & \text{if } Y \geq -\frac{a\sigma^2}{2} \\ \frac{Y_1+Y_2}{2}, & \text{if } -\sqrt{2}c\sigma \leq Y < -\frac{a\sigma^2}{2} \\ Y_2, & \text{if } Y < -\sqrt{2}c\sigma, \end{cases}$$

for $a \geq 2\sqrt{2}\frac{c}{\sigma}$,

$$\delta_{3,c,a}^*(\underline{Y}) = \begin{cases} \frac{Y_1+Y_2}{2} - \frac{a\sigma^2}{4}, & \text{if } Y \geq -\frac{a\sigma^2}{2} \\ Y_2, & \text{if } Y < -\frac{a\sigma^2}{2}. \end{cases}$$

Remark: (i) *It can be seen that the results of Theorem 3.1 fail to find an estimator dominating the generalized Bayes estimator $\delta_\eta(\underline{Y})$, given by (4.4). The question of admissibility or inadmissibility of the generalized Bayes estimator is unresolved.*

(ii) *On using Theorem 3.1, it can be shown that for $a < 0$, estimators $\delta_c(.)$, for $c < -\frac{a\sigma^2}{4}$, are inadmissible. Also, for $a > 0$, estimators $\delta_c(.)$, for $c \neq c_0$, are inadmissible.*

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