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LINEX Loss Functions with Applications to Determining the Optimum Process Parameters

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Abstract. In the classical Taguchi quality model, the symmetric quadratic loss function has been used to measure the loss of quality. However, there are a number of situations in which the symmetric quadratic loss may be inappropriate. In this paper, we proposed an asymmetric loss function, called linear exponential (LINEX) loss function, to determine optimum process parameters for the product quality. When the coefficient of LINEX loss function is small, it will be close to the quadratic loss. Moreover, the trade-off problem between quality and cost will be discussed.

Key words: LINEX loss function, trade-off problem, cost, quality.

1. Introduction

Specification limits (when they are, in fact, true tolerance limits) have generally been regarded as providing a range for the values of a process characteristic such that values within this range are acceptable. Implicit in this view is the idea that all values within this range are equally good. Common sense should tell us, however, that if we have a complicated piece of machinery that consists of a large number of moving parts, the machinery is likely to perform better if the dimensions of the individual part were made to conform to certain "optimal" values, than if the dimensions were merely within tolerance limits. This type of thinking is at the heart of the loss-function approach advocated by Taguchi, who contends that the "loss to society" increase as that value of a quality characteristic departs from its optimal value, regardless of whether or not a tolerance limit has been exceeded. The "loss to society" idea of Taguchi has been replaced by "long-term loss to the firm" by those in the Western world, but the general idea is the same. That is, individual firms and society as a whole suffer a loss when products do not function as they could if they were made properly.

The simplest type of loss function is squared error, which is also referred to as quadratic loss. Specifically, if we let y_0 denote a target value (i.e., optimal value) of a quality characteristic, y the actual value of that characteristic and L_y the loss that is incurred when $y \neq y_0$, then

$$L_y = (y - y_0)^2$$

would be a simple (quadratic) loss function. There are a number of reasons why it is often considered in evaluating quality characteristic. A major reason for the popularity of squared-error loss is due to its relationship to classical least squares theory.

A more general type of quadratic loss function (cf. Taguchi, 1986) is

$$L_y = k(y - y_0)^2, \tag{1}$$

where $k > 0$ is a constant that would have to be determined. Although the predicted loss at various of y is of interest, a practitioner may be more interested in the average squared-error loss, which is generally referred to as the mean squared error (MSE). For quadratic loss given by Equation (1), MSE(y) is given by

$$\mathrm{MSE}(y) = E(L_y) = kE(y - y_0)^2,$$

where E represents "expected value" (i.e., average).

An asymmetric loss function would be appropriate if the loss differs for values of y that are equidistant from the target. For example, a value that exceeds the target might be more detrimental than a value that is below the target. In that case, one could use the following loss functions (cf. Chen and Chou, 2004)

$$L_y = \begin{cases} k_1(y - y_0)^2 & \text{if } y \leq y_0, \\ k_2(y - y_0)^2 & \text{if } y > y_0, \end{cases} \quad \text{or} \quad L_y = \begin{cases} k_1(y_0 - y) & \text{if } y \leq y_0, \\ k_2(y - y_0) & \text{if } y > y_0, \end{cases}$$

where $k_2 > k_1 > 0$.

In this paper, we considered Varian's asymmetric LINEX loss function (cf. Varian, 1975; Zellner, 1986) that rises approximately exponentially on one side of zero and approximately linearly on the other side in his applied study of real estate assessment. Underassessment results in an approximately linear loss of revenue whereas overassessment often results in appeals with attendant, substantial litigation and other costs (cf. Varian, 1975). The LINEX loss function is given by

$$L_y = \exp(\phi(y - y_0)) - \phi(y - y_0) - 1, \quad \phi \neq 0. \tag{2}$$

In Figure 1, values of L_y are plotted in selected values of ϕ. It is seen that, for $\phi = 1$, the function is quite asymmetric with a value exceeding the target being more serious than a value below the target. But, for $\phi = -1$, the function is also quite asymmetric with a value below the target value being more serious than a value exceeding the target. Furthermore, when $\phi < 0$,

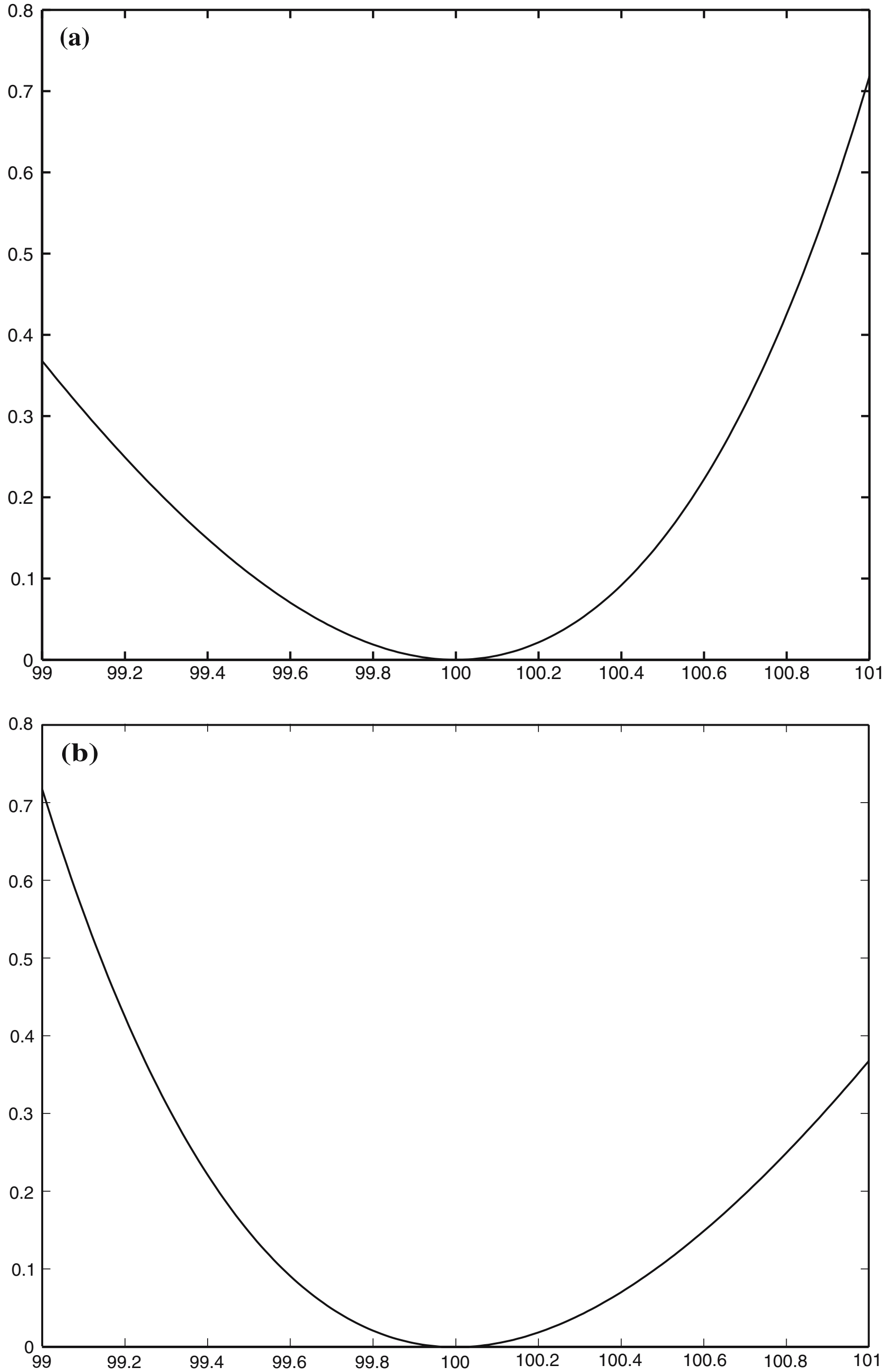

Figure 1. LINEX loss function with $y_0 = 100$ and (a) $\phi = 1$ (b) $\phi = -1$.

LINEX loss (2) rises almost exponentially when $y - y_0 < 0$ and almost linearly when $y - y_0 > 0$. For small value of ϕ, the LINEX loss can be expanded by Taylor's series

$$\begin{aligned}\exp(\phi(y-y_0)) - \phi(y-y_0) - 1 &= \sum_{i=0}^{\infty} \frac{\phi^i (y-y_0)^i}{i!} - \phi(y-y_0) - 1 \\ &= \sum_{i=2}^{\infty} \frac{\phi^i (y-y_0)^i}{i!} \\ &\approx \phi^2 (y-y_0)^2/2. \end{aligned} \tag{3}$$

Thus the LINEX loss is approximate to a quadratic loss (see Figure 2, $\phi = 0.1$).

The remainder of this paper is organized as follows. In Section 2, we will find the optimum parameters under LINEX loss function (2). In Section 3, the problem of trade-off between quality and cost (cf. Huang, 2001; Chen and Chou, 2004) under LINEX loss will be discussed. Numerical examples are given and comparisons are made between Huang (2001) and Chen and Chou (2004) in Section 4. Conclusions will be stated in Section 5.

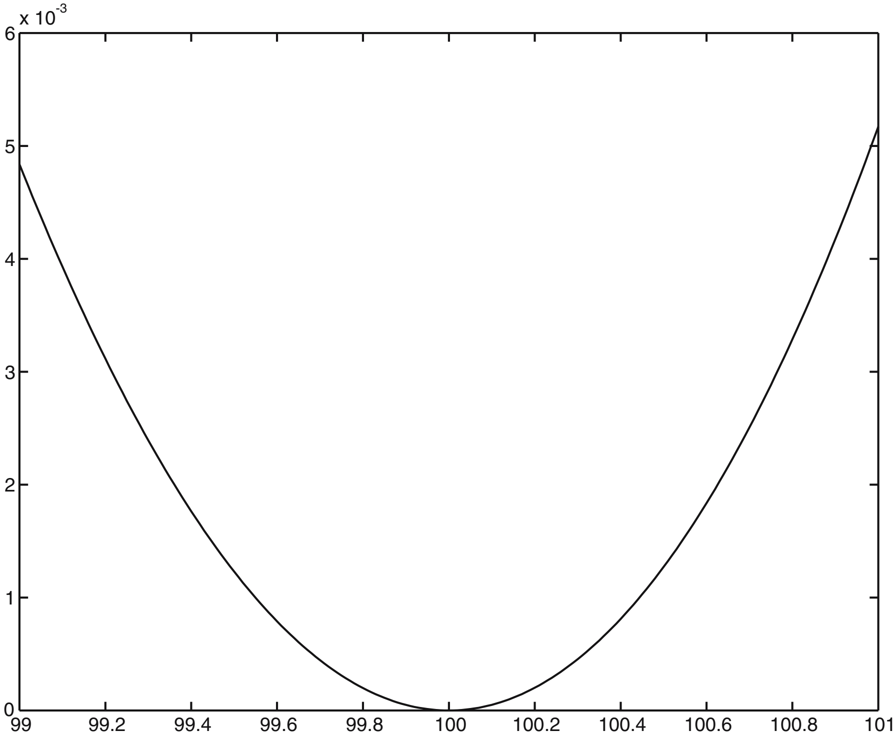

Figure 2. LINEX loss function with $y_0 = 100$ and $\phi = 0.1$.

2. The Optimum Parameters Minimize the LINEX Loss of Quality

Suppose that the input x is a normal random variable with mean μ and variance σ^2, that is, $x \sim N(\mu, \sigma^2)$. As the discussion of Huang (2001), the most useful models describing the input and output are the linear and quadratic ones. That is,

$$y = bx + c \quad \text{or} \quad y = ax^2 + bx + c,$$

where a, b and c are constants. Under LINEX loss function, the expected loss of quadratic one may be infinity. Thus, we only consider the output is a linear function of input.

It is well known that the moment generated function of x is

$$M_x(t) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right), \quad -\infty < t < \infty. \tag{4}$$

Therefore, we obtain $\mathrm{MSE}(y) = E(L_y)$ as follows:

$$\begin{aligned} E(L_y) &= kE[\exp(\phi(y - y_0)) - \phi(y - y_0) - 1] \\ &= k\{E[\exp(\phi y)]\exp(-\phi y_0) - \phi(b\mu + c) + \phi y_0 - 1\} \\ &= k\{E[\exp(\phi(bx + c))]\exp(-\phi y_0) - \phi(b\mu + c - y_0) - 1\} \\ &= k\{M_x(b\phi)\exp(\phi(c - y_0)) - \phi(b\mu + c - y_0) - 1\} \\ &= k\left\{\exp\left(\phi b\mu + \frac{\sigma^2 b^2 \phi^2}{2} + \phi(c - y_0)\right) - \phi(b\mu + c - y_0) - 1\right\}. \end{aligned}$$

In order to obtain the optimal parameter, $\tilde{\mu}$, we compute the partial derivative

$$\frac{\partial E(L_y)}{\partial \mu} = k\phi b\left\{\exp\left(\phi b\mu + \frac{\sigma^2 \phi^2 b^2}{2} + \phi(c - y_0)\right) - 1\right\}.$$

Thus, we have

$$\exp\left(\phi b\mu + \frac{\sigma^2 \phi^2 b^2}{2} + \phi(c - y_0)\right) - 1 = 0.$$

This implies

$$\phi b\mu + \frac{\sigma^2 \phi^2 b^2}{2} + \phi(c - y_0) = 0.$$

Hence

$$\tilde{\mu} = \frac{y_0 - c}{b} - \frac{\sigma^2 \phi b}{2}.$$

The second derivatives is $\dfrac{\partial^2 L_y}{\partial\mu^2} = k\phi^2 b^2 \exp(\phi b\mu + \dfrac{\sigma^2\phi^2 b^2}{2} + \phi(c - y_0)) > 0$. It means that $\tilde{\mu}$ minimizes $E(L_y)$, if σ is fixed.

Note that, when $\phi \to 0$, we have $\mu \to \dfrac{y_0 - c}{b}$, the best mean under quadratic loss.

3. The Trade-off Problem

In this section, we generalize Huang's cost model (Huang, 2001) that considers the trade-off problem between quality and cost. Assume the cost function is

$$C_x(\mu, \sigma) = \beta_1|\mu|^r + \frac{\beta_2}{\sigma^s}, \quad \beta_1, \beta_2 > 0, \ \ r \geq 1, \ \ s > 0.$$

According to Huang's assumption, the loss of profit is proportional to the loss of quality. That is, the loss of profit is

$$L_p = \overline{\alpha} L_y = \overline{\alpha} k[\exp(\phi(y - y_0)) - \phi(y - y_0) - 1].$$

Setting $\overline{\alpha} k \equiv \alpha$,

$$L_p = \alpha[\exp(\phi(y - y_0)) - \phi(y - y_0) - 1],$$

where α is call the coefficient of loss of profit due to the loss of quality. Thus the sum of the cost and the loss of profit is

$$\begin{aligned} T(\mu, \sigma) &= C_x(\mu, \sigma) + L_p \\ &= \beta_1|\mu|^r + \frac{\beta_2}{\sigma^s} + \alpha[\exp(\phi(y - y_0)) - \phi(y - y_0) - 1]. \end{aligned} \tag{5}$$

Now, we shall find the best mean $\tilde{\mu}$ and the best standard deviation $\tilde{\sigma}$ that minimize the average cost and average loss of profit denoted as $E[T(\mu, \sigma)]$.

For $x \sim N(\mu, \sigma^2)$, $E[T(\mu, \sigma)]$ is given by

$$\begin{aligned} E[T(\mu, \sigma)] = \beta_1|\mu|^r + \frac{\beta_2}{\sigma^s} + \alpha \Bigg\{ &\exp\left(\phi b\mu + \frac{\sigma^2 b^2 \phi^2}{2} + \phi(c - y_0)\right) \\ &- \phi(b\mu + c - y_0) - 1 \Bigg\}. \end{aligned}$$

Next, we will find $\tilde{\mu}$ and $\tilde{\sigma}$ by the following cases:

Case 1. If $\mu > 0$ and $r \neq 1$.

In this case

$$E[T(\mu,\sigma)] = \beta_1\mu^r + \frac{\beta_2}{\sigma^s} + \alpha\left\{\exp\left(\phi b\mu + \frac{\sigma^2 b^2\phi^2}{2} + \phi(c-y_0)\right)\right.$$
$$\left. -\phi(b\mu + c - y_0) - 1\right\}.$$

Then, the partial derivatives are

$$\frac{\partial E[T(\mu,\sigma)]}{\partial\mu} = r\beta_1\mu^{r-1} + \alpha\phi b\left\{\exp\left(\phi b\mu + \frac{\sigma^2\phi^2 b^2}{2} + \phi(c-y_0)\right) - 1\right\},$$
$$\frac{\partial E[T(\mu,\sigma)]}{\partial\sigma} = -\frac{s\beta_2}{\sigma^{s+1}} + \alpha\phi^2 b^2\sigma\exp\left(\phi b\mu + \phi(c-y_0) + \frac{\phi^2 b^2\sigma^2}{2}\right).$$

Therefore, $\tilde{\mu}$ and $\tilde{\sigma}$ must satisfy

$$r\beta_1\tilde{\mu}^{r-1} + \alpha\phi b\left\{\exp\left(\phi b\tilde{\mu} + \frac{\tilde{\sigma}^2\phi^2 b^2}{2} + \phi(c-y_0)\right) - 1\right\} = 0, \tag{6}$$

$$-\frac{s\beta_2}{\tilde{\sigma}^{s+1}} + \alpha\phi^2 b^2\tilde{\sigma}\exp\left(\phi b\tilde{\mu} + \phi(c-y_0) + \frac{\phi^2 b^2\tilde{\sigma}^2}{2}\right) = 0. \tag{7}$$

Using (7), we have

$$\exp\left(\phi b\tilde{\mu} + \phi(c-y_0) + \frac{\phi^2 b^2\tilde{\sigma}^2}{2}\right) = \frac{s\beta_2}{\tilde{\sigma}^{s+2}\alpha\phi^2 b^2}. \tag{8}$$

Substituting (8) in (6), we obtain

$$r\beta_1\tilde{\mu}^{r-1} + \frac{s\beta_2}{\tilde{\sigma}^{s+2}\phi b} - \alpha\phi b = 0.$$

This implies

$$\tilde{\mu} = \left[\frac{\alpha\phi b - (s\beta_2/\tilde{\sigma}^{s+2}\phi b)}{r\beta_1}\right]^{1/r-1} \tag{9}$$

and

$$\exp\left(\phi b\left[\frac{\alpha\phi b - (s\beta_2/\tilde{\sigma}^{s+2}\phi b)}{r\beta_1}\right]^{1/r-1} + \phi(c-y_0) + \frac{\phi^2 b^2\tilde{\sigma}^2}{2}\right) = \frac{s\beta_2}{\tilde{\sigma}^{s+2}\alpha\phi^2 b^2}. \tag{10}$$

Thus, we can find $\tilde{\sigma}$ by solving Equation (10) and then substitute $\tilde{\sigma}$ in (9) to find $\tilde{\mu}$. The unique root of (10) exists only when $\tilde{\sigma}^{s+2} > \frac{s\beta_2}{\alpha\phi^2 b^2}$. Note that the second derivatives are

$$\frac{\partial^2 E[T(\mu,\sigma)]}{\partial\mu^2} = r(r-1)\beta_1\mu^{r-2} + \alpha\phi^2 b^2 \exp\left(\phi b\mu + \frac{\sigma^2\phi^2 b^2}{2} + \phi(c-y_0)\right),$$

$$\frac{\partial^2 E[T(\mu,\sigma)]}{\partial\sigma^2} = \frac{s(s+1)\beta_2}{\sigma^{s+2}} + \alpha\phi^2 b^2(1+\phi^2 b^2\sigma^2)\exp\left(\phi b\mu + \phi(c-y_0) + \frac{\phi^2 b^2\sigma^2}{2}\right)$$

and

$$\frac{\partial^2 E[T(\mu,\sigma)]}{\partial\mu\partial\sigma} = \alpha\phi^3 b^3\sigma \exp\left(\phi b\mu + \phi(c-y_0) + \frac{\phi^2 b^2\sigma^2}{2}\right)$$

It follows that the matrix

$$\begin{bmatrix} \frac{\partial^2 E[T(\mu,\sigma)]}{\partial\mu^2} & \frac{\partial^2 E[T(\mu,\sigma)]}{\partial\sigma\partial\mu} \\ \frac{\partial^2 E[T(\mu,\sigma)]}{\partial\mu\partial\sigma} & \frac{\partial^2 E[T(\mu,\sigma)]}{\partial\sigma^2} \end{bmatrix}$$

is positive definite. That is, $\tilde{\mu}$ and $\tilde{\sigma}$ are the best solutions.

Case 2. If $\mu < 0, \quad r \neq 1$.
In this case

$$E[T(\mu,\sigma)] = \beta_1(-\mu)^r + \frac{\beta_2}{\sigma^s} + \alpha\left\{\exp\left(\phi b\mu + \frac{\sigma^2 b^2\phi^2}{2} + \phi(c-y_0)\right) - \phi(b\mu + c - y_0) - 1\right\}.$$

$$\frac{\partial E[T(\mu,\sigma)]}{\partial\mu} = -r\beta_1(-\mu)^{r-1} + \alpha\phi b\left\{\exp\left(\phi b\mu + \frac{\sigma^2\phi^2 b^2}{2} + \phi(c-y_0)\right) - 1\right\},$$

$$\frac{\partial E[T(\mu,\sigma)]}{\partial\sigma} = -\frac{s\beta_2}{\sigma^{s+1}} + \alpha\phi^2 b^2\sigma \exp\left(\phi b\mu + \phi(c-y_0) + \frac{\phi^2 b^2\sigma^2}{2}\right).$$

Similarly, $\tilde{\mu}$ and $\tilde{\sigma}$ must satisfy

$$\tilde{\mu} = -\left[\frac{(s\beta_2/\tilde{\sigma}^{s+2}\phi b) - \alpha\phi b}{r\beta_1}\right]^{1/r-1}, \tag{11}$$

$$\exp\left(-\phi b\left[\frac{(s\beta_2/\tilde{\sigma}^{s+2}\phi b)-\alpha\phi b}{r\beta_1}\right]^{1/r-1}+\phi(c-y_0)+\frac{\phi^2 b^2\tilde{\sigma}^2}{2}\right)=\frac{s\beta_2}{\tilde{\sigma}^{s+2}\alpha\phi^2 b^2}. \tag{12}$$

The unique root of (12) exists only when $\tilde{\sigma}^{s+2}<\dfrac{s\beta_2}{\alpha\phi^2 b^2}$.

Case 3. If $\mu>0, \quad r=1$.
In this case

$$\begin{aligned} E[T(\mu,\sigma)] &= \beta_1\mu+\frac{\beta_2}{\sigma^s}+\alpha\left\{\exp\left(\phi b\mu+\frac{\sigma^2 b^2\phi^2}{2}+\phi(c-y_0)\right)\right. \\ &\qquad \left.-\phi(b\mu+c-y_0)-1\right\}. \\ \frac{\partial E[T(\mu,\sigma)\}}{\partial\mu} &= \beta_1+\alpha\phi b\left\{\exp\left(\phi b\mu+\frac{\sigma^2\phi^2 b^2}{2}+\phi(c-y_0)\right)-1\right\}, \\ \frac{\partial E[T(\mu,\sigma)]}{\partial\sigma} &= -\frac{s\beta_2}{\sigma^{s+1}}+\alpha\phi^2 b^2\sigma\exp\left(\phi b\mu+\phi(c-y_0)+\frac{\phi^2 b^2\sigma^2}{2}\right), \end{aligned}$$

$\tilde{\mu}$ and $\tilde{\sigma}$ must satisfy

$$\begin{aligned} \tilde{\sigma} &= \left[\frac{s\beta_2}{\phi b(\alpha\phi b-\beta_1)}\right]^{1/s+2} \quad \text{and} \\ \tilde{\mu} &= \left[\phi(y_0-c)-\frac{\phi^2 b^2\tilde{\sigma}^2}{2}+\ln\frac{\alpha\phi b-\beta_1}{\alpha\phi b}\right]\Big/\phi b. \end{aligned}$$

Case 4. If $\mu<0, \quad r=1$.

In this case

$$\begin{aligned} E[T(\mu,\sigma)] &= -\beta_1\mu+\frac{\beta_2}{\sigma^s}+\alpha\left\{\exp\left(\phi b\mu+\frac{\sigma^2 b^2\phi^2}{2}+\phi(c-y_0)\right)\right. \\ &\qquad \left.-\phi(b\mu+c-y_0)-1\right\}. \\ \frac{\partial E[T(\mu,\sigma)]}{\partial\mu} &= -\beta_1+\alpha\phi b\left\{\exp\left(\phi b\mu+\frac{\sigma^2\phi^2 b^2}{2}+\phi(c-y_0)\right)-1\right\}, \\ \frac{\partial E[T(\mu,\sigma)]}{\partial\sigma} &= -\frac{s\beta_2}{\sigma^{s+1}}+\alpha\phi^2 b^2\sigma\exp\left(\phi b\mu+\phi(c-y_0)+\frac{\phi^2 b^2\sigma^2}{2}\right), \end{aligned}$$

$\tilde{\mu}$ and $\tilde{\sigma}$ must satisfy

$$\tilde{\sigma} = \left[\frac{s\beta_2}{\phi b(\alpha\phi b + \beta_1)}\right]^{1/s+2} \quad \text{and}$$
$$\tilde{\mu} = \left[\phi(y_0 - c) - \frac{\phi^2 b^2 \tilde{\sigma}^2}{2} + \ln\frac{\alpha\phi b + \beta_1}{\alpha\phi b}\right] \Big/ \phi b.$$

Case 5. If $\mu = 0$.
In this case

$$E[T(0,\sigma)] = \frac{\beta_2}{\sigma^s} + \alpha\left\{\exp\left(\frac{\sigma^2 b^2 \phi^2}{2} + \phi(c - y_0)\right) - \phi(c - y_0) - 1\right\}.$$
$$\frac{\partial E[T(0,\sigma)]}{\partial\sigma} = -\frac{s\beta_2}{\sigma^{s+1}} + \alpha\phi^2 b^2 \sigma \exp\left(\phi(c - y_0) + \frac{\phi^2 b^2 \sigma^2}{2}\right).$$

Then we have

$$\tilde{\sigma} = \left\{2\left[\ln\left(\frac{s\beta_2}{\sigma^{s+2}\alpha\phi^2 b^2}\right) - \phi(c - y_0)\right]\right\}^{1/2} \Big/ \phi b.$$

4. Numerical Examples

In this section, we consider a numerical example to illustrate our result. As the Example 1 in Huang (2001), set $y_0 = 100$ and assume $C_x(\mu,\sigma) = 2\mu^2 + 1/\sigma^2$, $y = 5x + 50$ and $\alpha = 5$. The optimal solution of Huang (2001) is $(\tilde{\mu},\tilde{\sigma})$=(9.82, 0.3) with $E[T(\tilde{\mu},\tilde{\sigma})]$=219.21.

Chen and Chou (2004) also discussed this. Under the asymmetric loss situation, they set $\bar{\alpha} = 0.1$, $k_1 = 50$ and $k_2 = 100$. The optimum solution of asymmetric quadratic quality loss in Section 1 is $(\tilde{\mu},\tilde{\sigma}) = (9.89, 0.28)$ with $E[T(\tilde{\mu},\tilde{\sigma})] = 225.8541$; the optimum solution of asymmetric linear quality loss is $(\tilde{\mu},\tilde{\sigma}) = (6.19, 1.4)$ with $E[T(\tilde{\mu},\tilde{\sigma})] = 172.4939$.

Considering LINEX loss function, we set $\phi = 1$. Then Equation (10) can be rewritten as

$$\exp\left(5\left[\frac{25 - (2/5\tilde{\sigma}^4)}{4}\right] - 50 + \frac{25\tilde{\sigma}^2}{2}\right) = \frac{2}{125\tilde{\sigma}^4}.$$

It implies

$$\frac{-1}{2\tilde{\sigma}^4} - \frac{75}{4} + \frac{25\tilde{\sigma}^2}{2} - \ln\frac{2}{125\tilde{\sigma}^4} = 0.$$

Thus we have $\tilde{\sigma} = 1.083$. By Equation (9), we have

$$\tilde{\mu} = \frac{25 - 2/(5 \times 1.083^4)}{4} = 6.177.$$

Therefore, $E[T(\tilde{\mu},\tilde{\sigma})]=167.80$.

Based on Equations (3) and (5), we see that if α is proportional to ϕ^{-2} then the optimal solution of LINEX loss will be close to quadratic loss when ϕ is small. In the next, we set $\phi=0.1$ and $\alpha=500$. Similarly, (10) can be rewritten as

$$\exp\left(0.5\left[\frac{250-(2/0.5\tilde{\sigma}^4)}{4}\right]-5+\frac{0.25\tilde{\sigma}^2}{2}\right)=\frac{2}{125\tilde{\sigma}^4}.$$

It implies

$$\frac{-1}{2\tilde{\sigma}^4}+\frac{105}{4}+\frac{0.25\tilde{\sigma}^2}{2}-\ln\frac{2}{125\tilde{\sigma}^4}=0.$$

Thus, we have $(\tilde{\mu},\tilde{\sigma})=(9.716,0.371)$ with $E[T(\tilde{\mu},\tilde{\sigma})]=208.4$. Furthermore, we find this value is close to the optimal solution of Huang (2001).

5. Conclusions

In this paper we use the LINEX loss functions in Taguchi quality model and Huang's cost model. Theoretically, if α is proportional to ϕ^{-2} then our model will close to Huang's cost model when ϕ is closed to zero. Thus Huang's cost model is a special case of our model. The paper also gives numerical examples for illustrated. On the other hand, the LINEX loss function need to determine the value of ϕ. But, the asymmetric loss function proposed by Chen and Chou (2004) need to determine the values of k_1 and k_2. Furthermore, the LINEX loss functions are more flexible than Chen and Chou's asymmetric loss functions. Therefore, the LINEX loss functions are well suited to use in Taguchi quality model.

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