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Optimal process parameters under LINEX loss function with general input quality characteristic

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Abstract In this paper we generalize the quality and cost trade-off problem of Chang and Hung (Qual Quant 41: 291–301, 2007) under the LINEX loss function. We consider the general input characteristic given by the random variable X with moment generating function $m_X(t)$ and output characteristic given by the deterministic transformation $Y = g(X)$. The two cases we consider are when $g(X)$ is an affine function of X and X follows (1) the gamma distribution, and (2) the double exponential distribution.

Keywords Asymmetric loss function · Taguchi quality model · Gamma distribution · Double exponential distribution · Laplace distribution

1 Introduction

In this paper we generalize the work of Chang and Hung (2007) who examine the trade-off problem between quality and cost defined by Huang (2001). Huang defines the trade-off between quality and cost as an extension of the classical Taguchi quality model. In this model the firm's total quality cost includes a financial loss due to the loss in quality (the Taguchi loss) and the cost to control the mean and standard deviation of the input quality characteristic. In the classical Taguchi model, the quality loss is measured using the symmetric square error loss function. Chen and Chou (2004) generalize Huang's trade-off problem between quality and cost by introducing asymmetric loss functions. Chen and Chou consider the asymmetric quadratic quality loss function and the asymmetric linear quality loss function. Chang and Hung have further generalized the work of Chen and Chou by considering the trade-off problem between quality and cost under the asymmetric LINEX (LINear EXponential) loss function of Varian (1975). The LINEX loss function has received considerable attention over

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the years. Some basic properties of the LINEX loss function and the effects of asymmetry on some well-known statistical models are discussed in Zellner (1986).

Chang and Hung (2007) consider the trade-off problem between quality and cost for the case when the input quality characteristic is normally distributed. In this paper we will generalize that result and show that the input quality characteristic may have any distribution for which a moment generating function exists provided that the input–output relationship is an affine function. However, some distributions are analytically and numerically more tractable than others. As examples we consider input quality characteristics from the gamma distribution and the double exponential distribution. These distributions are of particular interest because they provide alternatives to the log-normal distribution and the normal distribution, respectively. The gamma distribution provides a flexible family of distributions which contains other well-known distributions such as the chi-squared distribution and the exponential distribution as members. The gamma distribution can be parameterized to take many shapes to describe the stochastic behavior of non-negative random variables. The double exponential distribution provides an alternative to the symmetric normal distribution which has heavier tails than the normal distribution.

The remainder of this paper is organized as follows. In Sect. 2 we discuss quality loss under the LINEX loss function and some potential difficulties and restrictions in using the LINEX loss function to measure quality loss. In particular we will address the quality loss problem when the input quality characteristic has a gamma distribution and a double exponential distribution. In Sect. 3 we will consider the corresponding trade-off problems for the input quality characteristics in the case of the gamma distribution and the double exponential distribution with numerical examples. We conclude the paper in Sect. 4.

2 Quality loss under the LINEX loss function

We follow the standard notational conventions and let X and Y represent random variables and x and y represent realizations of these random variables. The input quality characteristic X is related to the output quality characteristic Y by the transformation $Y = g(X)$. The function $g(X)$ is commonly taken to be a polynomial function of X. We define the polynomial function $p_n(X)$ as

$$p_n(X) = a_0 + a_1 X + \cdots + a_n X^n. \tag{1}$$

The two particular polynomial functions typically assumed in the literature are the affine function

$$p_1(X) = a_0 + a_1 X \tag{2}$$

and the quadratic function

$$p_2(X) = a_0 + a_1 X + a_2 X^2. \tag{3}$$

We denote the target value of the output quality characteristic Y by y_0. The loss $L_{y_0}(Y)$ is a measure of the deviance of Y from the desired target value y_0. Under the LINEX loss specification, this loss is given by

$$L_{y_0}(Y) = k\left(\exp(\phi(Y - y_0)) - \phi(Y - y_0) - 1\right) \tag{4}$$

for $\phi \neq 0$ and $k > 0$. The sign of ϕ determines whether over-estimates or under-estimates are more heavily penalized. If $\phi > 0$, the LINEX loss function penalizes over-estimation more heavily, the term $\exp(\phi(Y - y_0)) \to \infty$ as $(Y - y_0) \to \infty$. If $\phi < 0$, the LINEX loss function penalizes under-estimation more heavily, the term $\exp(\phi(Y - y_0)) \to 0$ as $(Y - y_0) \to \infty$.

In the quality loss problem we are interested in selecting the optimal process parameters to minimize the expected loss $\mathbb{E}[L_{y_0}(Y)]$. By computing the expected loss under the LINEX loss function, we can see the limitation in the input–output transformations of the quality characteristics we can consider.

$$\begin{aligned}
\mathbb{E}[L_{y_0}(Y)] &= \mathbb{E}\left[k\left(\exp(\phi(Y - y_0)) - \phi(Y - y_0) - 1\right)\right] && (5)\\
&= k\mathbb{E}\left[\exp(\phi Y)\exp(-\phi y_0) - \phi Y + \phi y_0 - 1\right] && (6)\\
&= k\left(\mathbb{E}[\exp(\phi Y)]\exp(-\phi y_0) - \phi\mathbb{E}[Y] + \phi y_0 - 1\right) && (7)
\end{aligned}$$

Different specifications of the input–output relationship $Y = g(X)$ make $\mathbb{E}[\exp(\phi Y)]$ easy, difficult, or impossible to compute analytically. The LINEX loss function is well suited when the input–output relationship is an affine transformation. The affine function $Y = a_0 + a_1 X$ yields

$$\begin{aligned}
\mathbb{E}[\exp(\phi Y)] &= \mathbb{E}[\exp(\phi(a_0 + a_1 X))] && (8)\\
&= \exp(a_0\phi)\mathbb{E}[\exp(\phi a_1 X)] && (9)
\end{aligned}$$

where $\mathbb{E}[\exp(\phi a_1 X)]$ is the moment generating function of X defined by $m_X(t) = \mathbb{E}[\exp(tX)]$ with $t = \phi a_1$. In this case we can rewrite the expected loss in a more convenient and general form which we will reference throughout the paper.

$$\mathbb{E}[L_{y_0}(Y)] = k\left(\exp\left(\phi(a_0 - y_0)\right) m_X(\phi a_1) - \phi a_1\mathbb{E}[X] - \phi(a_0 - y_0) - 1\right) \quad (10)$$

Other specifications for $Y = g(X)$ are not as mathematically tractable. We are not guaranteed that $\mathbb{E}[\exp(\phi Y)]$ will exist in cases where $Y = g(X)$ is a quadratic (or higher order) polynomial. When $g(X)$ is affine, the LINEX loss function has obtained its maximum mathematical tractability. In this case we can consider input quality characteristics from any distribution for which the moment generating function exists. However, this restriction prohibits the use of several popular continuous distributions such as the log-normal distribution, the Pareto distribution, and the t-distribution as input quality characteristics.

Chang and Hung (2007) consider the case when $g(X)$ is an affine transformation of the $N(\mu, \sigma^2)$ random variable X. As a generalization of their work, we will consider the affine transformation when X follows the gamma distribution and when X follows the double exponential distribution. As mentioned in the introduction, the gamma distribution is a flexible family of distributions for which the moment generating function exists and which contains the chi-squared distribution and the exponential distribution as members. The gamma distribution can be parameterized to approximate the log-normal distribution, which has no moment generating function, and other distributions restricted to positive values for the support. The gamma distribution is a viable and useful distribution to consider in connection with the LINEX loss function. The double exponential distribution is another distribution for which the moment generating function exists. It is a symmetric distribution which provides an alternative to the normal distribution and the t-distribution, which has no moment generating function.

2.1 The gamma input quality characteristic

We first consider the case where $g(X)$ is an affine function and X is a Gamma(α,β) random variable. We use the typical parameterization of the gamma probability density function (pdf). Since several parameterizations exist, we explicitly state the parameterization for the reader's convenience. The pdf of a Gamma(α,β) random variable will be given by

$$f(x) = \frac{1}{\beta^{\alpha}\Gamma(\alpha)} x^{\alpha-1} \exp\left(\frac{-x}{\beta}\right), \quad x > 0 \tag{11}$$

for $\alpha, \beta > 0$. The parameter α is called the shape parameter, β is called the scale parameter, and $\Gamma(\alpha)$ is the typical gamma function defined by the integral

$$\Gamma(\alpha) = \int_0^{\infty} x^{\alpha-1} \exp(-x) dx \tag{12}$$

for $\alpha > 0$. In this parameterization the mean is given by $\mathbb{E}[X] = \alpha\beta$ and the variance is given by $\mathbb{V}[X] = \alpha\beta^2$. The moment generating function of a gamma random variable exists, and it is given by

$$m_X(t) = \left(\frac{1}{1-\beta t}\right)^{\alpha}. \tag{13}$$

Substituting the moment generating function and the mean for a gamma random variable into Eq. 10, the expected loss of quality for a gamma distributed input quality characteristic is given by

$$\mathbb{E}[L_{y_0}(Y)] = k\left(\exp(\Lambda)\left(\frac{1}{\Psi}\right)^{\alpha} - \phi a_1(\alpha\beta) - \Lambda - 1\right) \tag{14}$$

where $\Lambda = \phi(a_0 - y_0)$ and $\Psi = 1 - a_1\phi\beta$. The presence of α as an exponent does not yield an analytical solution. This function will have to be minimized numerically in order to solve for the optimal parameters α^* and β^*. However, we can derive an analytical sufficient condition for a local minimum for the given system parameters. We will first compute the first partial derivatives with respect to α and β. We remind the reader that $d/dx\left[a^x\right] = a^x \ln(a)$ for a constant a.

$$\frac{\partial}{\partial\alpha}\mathbb{E}[L_{y_0}(Y)] = -\,k\exp(\Lambda)\Psi^{-\alpha}\ln(\Psi) - ka_1\phi\beta \tag{15}$$

$$\frac{\partial}{\partial\beta}\mathbb{E}[L_{y_0}(Y)] = \alpha k a_1\phi\left[\exp(\Lambda)\Psi^{-(\alpha+1)} - 1\right] \tag{16}$$

The second partial derivatives are given by

$$\frac{\partial^2}{\partial\alpha^2}\mathbb{E}[L_{y_0}(Y)] = k\exp(\Lambda)\Psi^{-\alpha}\ln(\Psi)^2 \tag{17}$$

$$\frac{\partial^2}{\partial\beta\partial\alpha}\mathbb{E}[L_{y_0}(Y)] = ka_1\phi\left[\exp(\Lambda)\Psi^{-(\alpha+1)}(1-\alpha\ln(\Psi)) - 1\right] \tag{18}$$

$$\frac{\partial^2}{\partial\beta^2}\mathbb{E}[L_{y_0}(Y)] = \alpha(\alpha+1)k(a_1\phi)^2\exp(\Lambda)\Psi^{-(\alpha+2)}. \tag{19}$$

From the second partial derivatives the determinant of the Hessian matrix is given by

$$\begin{aligned}|\mathbf{H}| = (ka_1\phi)^2\exp(2\Lambda)\Psi^{-2(\alpha+1)}\Big[&\alpha(\alpha+1)\ln(\Psi)^2 - [1-\alpha\ln(\Psi)]^2 \\ &+ 2\Psi^{\alpha+1}[1-\alpha\ln(\Psi)] - \exp(-2\Lambda)\Psi^{2(\alpha+1)}\Big]\end{aligned} \tag{20}$$

where α and β are restricted to begin positive. After simplifying the determinant, a sufficient condition for the Hessian matrix to be positive definite is given by

$$\alpha \ln(\Psi)\left[\ln(\Psi) - 2\Psi^{\alpha+1} + 2\right] + \Psi^{\alpha+1}\left(2 - \exp(-2\Lambda)\Psi^{\alpha+1}\right) - 1 > 0 \tag{21}$$

for $\alpha = \alpha^*$ and $\beta = \beta^*$. Notice that we require $\Psi = 1 - a_1\phi\beta > 0$ in order for $\ln(\Psi)$ to be defined. This restriction requires that a_1 and ϕ be opposite in sign.

Analytically, we may consider a more mathematically tractable case by restricting the shape parameter to the fixed value α_0. In this case Eq. 14 is a function of β alone. We can solve Eq. 16 for β^*

$$\beta^* = \frac{1}{a_1\phi}\left(1 - \left(\frac{\exp(\phi(a_0 - y_0))}{\alpha_0 k a_1 \phi}\right)^{1/(\alpha_0+1)}\right), \tag{22}$$

and show that β^* is a well defined minimum. Substituting β^* into Eq. 19, we see that $\partial^2\mathbb{E}[L_{y_0}(Y)]/\partial\beta^2 > 0$ if $a_1\phi > 0$.

2.2 The double exponential input quality characteristic

We now consider the case where $g(X)$ is an affine function and X is a double exponential random variable. The double exponential distribution is a location-scale family with location parameter α and scale parameter β. We will denote the distribution by DE(α,β). The pdf for a DE(α,β) random variable is given by

$$f(x) = \frac{1}{2\beta}\exp\left(\frac{-|x-\alpha|}{\beta}\right), \quad -\infty < x < \infty \tag{23}$$

for $\alpha \in \mathbb{R}$ and $\beta > 0$. You may recognize this distribution by another name. The double exponential distribution is also known as the Laplace distribution. The mean is given by $\mathbb{E}[X] = \alpha$ and the variance is given by $\mathbb{V}[X] = 2\beta^2$. The moment generating function of a double exponential random variable exists, and it is given by

$$m_X(t) = \frac{\exp(\alpha t)}{1 - (\beta t)^2}. \tag{24}$$

If we substitute the mean and the moment generating function into Eq. 10, the expected loss of quality is given by

$$\mathbb{E}[L_{y_0}(Y)] = k\left[\exp(\Theta)\,\Delta^{-1} - \phi\Theta - 1\right] \tag{25}$$

where $\Theta = \phi(a_0 - y_0 + \alpha a_1)$ and $\Delta = 1 - (a_1\phi\beta)^2$. Once again, the expected loss will need to be minimized numerically to solve for the optimal parameters α^* and β^*. The first partial derivatives are given by

$$\frac{\partial}{\partial\alpha}\mathbb{E}[L_{y_0}(Y)] = k a_1 \phi\left[\exp(\Theta)\Delta^{-1} - 1\right] \tag{26}$$

$$\frac{\partial}{\partial\beta}\mathbb{E}[L_{y_0}(Y)] = 2\beta k (a_1\phi)^2\exp(\Theta)\Delta^{-2}. \tag{27}$$

We should remember that β is restricted to be positive. Finding a root of Eq. 27 may be difficult, and it my be better to apply standard techniques for the optimization of nonlinear functions (Dennis and Schnabel 1987) directly to Eq. 25. The second partial derivatives are given by

$$\frac{\partial^2}{\partial\alpha^2}\mathbb{E}[L_{y_0}(Y)] = k(a_1\phi)^2\exp(\Theta)\Delta^{-1} \tag{28}$$

$$\frac{\partial^2}{\partial\beta\partial\alpha}\mathbb{E}[L_{y_0}(Y)] = 2\beta k(a_1\phi)^3\exp(\Theta)\Delta^{-2} \tag{29}$$

$$\frac{\partial^2}{\partial\beta^2}\mathbb{E}[L_{y_0}(Y)] = 2k(a_1\phi)^2\exp(\Theta)\Delta^{-2}\left[(2\beta a_1\phi)^2\Delta^{-1}+1\right]. \tag{30}$$

The first and second partial derivatives require that $1-(a_1\phi\beta)^2 \neq 0$. The determinant of the Hessian matrix is given by

$$|\mathbf{H}| = 2k^2(a_1\phi)^4\exp(2\Theta)\Delta^{-3}\left[2(a_1\phi\beta)^2\Delta^{-1}+1\right]. \tag{31}$$

After simplifying the expression for the determinant of the Hessian matrix, a sufficient condition for the Hessian matrix to be positive definite is given by

$$1+(a_1\phi\beta)^2 > 0. \tag{32}$$

This condition is true for all values of a_1, ϕ, and β such that $\Delta = 1-(a_1\phi\beta)^2 \neq 0$.

Similar to the case of the gamma distribution, we can restrict the double exponential distribution to have a known scale parameter β_0. In this case we can solve Eq. 26 for α^*.

$$\alpha^* = \frac{1}{a_1}\left[\frac{1}{\phi}\ln\left(\frac{1}{ka_1\phi}[1-(a_1\phi\beta_0)^2]\right)+(y_0-a_0)\right] \tag{33}$$

The optimal solution α^* requires that $[1-(a_1\phi\beta_0)^2]/ka_1\phi > 0$ in order for $\ln\left(\frac{1}{ka_1\phi}[1-(a_1\phi\beta_0)^2]\right)$ to be defined. In addition Eq. 28 requires the restriction that $\Delta = \left(1-(a_1\phi\beta)^2\right)^{-1} > 0$, which seems highly restrictive.

While we are able to derive analytical expressions for the partial derivatives and the determinant of the Hessian matrix for both the gamma distribution and the double exponential distribution, the existence of solutions α^* and β^* strongly depends on the values of the system parameters. Most cases will need to be considered individually, but these general expressions have yielded some necessary conditions on the system parameters. In the following section we will combine the loss of quality with the cost to control the process mean and standard deviation. We will consider cases in which one the parameters α or β is fixed. In these cases we will find that the solutions are well behaved in practice.

3 The quality and cost trade-off problem

Following Chang and Hung (2007), the cost to control the mean μ and the standard deviation σ of the input quality characteristic X is given by

$$C_X(\mu,\sigma) = \kappa_1|\mu|^r + \frac{\kappa_2}{\sigma^s} \tag{34}$$

for $\kappa_1, \kappa_2 \geq 0$, $r \geq 1$, and $s > 0$.

We denote the loss of profit due to the loss of quality by $\tilde{\pi}$. Huang (2001) assumes that the loss of profit $\tilde{\pi}(Y)$ is proportional to the loss of quality, that is $\tilde{\pi}(Y) = \delta L_{y_0}(Y)$ where $\delta > 0$ is the coefficient of loss of profit due to loss of quality or the financial penalty for loss of quality. Under the LINEX loss function,

$$\begin{aligned}\tilde{\pi}(Y) &= \delta k \left[\exp(\phi(Y-y_0)) - \phi(Y-y_0) - 1\right] \\ &= \gamma\left[\exp(\phi(Y-y_0)) - \phi(Y-y_0) - 1\right]\end{aligned} \tag{35}$$

for $\gamma = \delta k > 0$.

The trade-off problem between quality and cost is defined by the total cost due to the loss of profit due to the loss of quality and the cost to control the mean and standard deviation of the input quality characteristic. The total cost function is given by

$$T(\mu,\sigma) = C_X(\mu,\sigma) + \tilde{\pi}(Y) \tag{36}$$

$$= \kappa_1|\mu|^r + \frac{\kappa_2}{\sigma^s} + \gamma\left[\exp(\phi(Y-y_0)) - \phi(Y-y_0) - 1\right]. \tag{37}$$

In general we cannot simultaneously minimize both $C_X(\mu,\sigma)$ and $\tilde{\pi}(Y)$, hence we have a trade-off problem. When Y is an affine function of the random variable X with mean μ, standard deviation σ, and moment generating function $m_X(\phi a_1)$, the expected total cost function is given by

$$\begin{aligned}\mathbb{E}[T(\mu,\sigma)] = \kappa_1|\mu|^r + \frac{\kappa_2}{\sigma^s} + \gamma\left[\exp\left(\phi(a_0-y_0)\right) m_X(\phi a_1)\right. \\ \left. -\ \phi a_1 \mathbb{E}[X] - \phi(a_0-y_0) - 1\right].\end{aligned} \tag{38}$$

Substituting the mean, variance, and moment generating function of the gamma random variable into Eq. 38, the expected total cost function for the quality and cost trade-off problem is given by

$$\begin{aligned}\mathbb{E}[T(\alpha,\beta)] = \kappa_1|\alpha\beta|^r + \frac{\kappa_2}{(\alpha\beta^2)^{s/2}} + \gamma\exp(\phi(a_0-y_0))(1-a_1\phi\beta)^{-\alpha} \\ -\gamma a_1\phi(\alpha\beta) - \gamma\left[\phi(a_0-y_0) - 1\right].\end{aligned} \tag{39}$$

The analogous substitutions for the double exponential random variable yields the expected total cost function

$$\begin{aligned}\mathbb{E}[T(\alpha,\beta)] = \kappa_1|\alpha|^r + \frac{\kappa_2}{(2\beta^2)^{s/2}} + \gamma\exp(\phi(a_0-y_0))\left(1-(a_1\phi\beta)^2\right)^{-1} \\ -\gamma a_1\phi\alpha - \gamma\left[\phi(a_0-y_0) - 1\right].\end{aligned} \tag{40}$$

Neither expected total cost functions in Eq. 39 nor Eq. 40 yield nice analytical solutions, as in the case of the normal input quality characteristic in Chang and Hung (2007). Instead, both of these expected total cost functions will have to be minimized numerically for given system parameters κ_1, κ_2, r, s, γ, ϕ, a_0, a_1, and y_0.

Both Eq. 39 and 40 indicate that the total cost functions cannot be treated as functions of two parameters. For the gamma random variable the parameters α and β in Eq. 39 are not identifiable since they appear together as $\alpha\beta$ and $\alpha\beta^2$. Heuristically, the total cost function would be minimized if $\alpha\beta \to 0$ and $\alpha\beta^2 \to \infty$. Similarly, for the double exponential distribution Eq. 40 would be minimized if $\alpha \to 0$ and $\beta \to \infty$.

In order to make the optimization problems better posed, we treat the shape parameter of the gamma distribution as a system parameter, $\alpha = \alpha_0$, and optimize over β. In this formulation the expected total cost function is now function of a single unknown, β.

$$\begin{aligned}\mathbb{E}[T(\beta)] = \kappa_1|\alpha_0\beta|^r + \frac{\kappa_2}{(\alpha_0\beta^2)^{s/2}} + \gamma\exp(\phi(a_0-y_0))(1-a_1\phi\beta)^{-\alpha_0} \\ -\gamma a_1\phi(\alpha_0\beta) - \gamma\left[\phi(a_0-y_0) - 1\right].\end{aligned} \tag{41}$$

Notice that we require that $1 - a_1\phi\beta > 0$ if $0 < \alpha_0 < 1$. The expected total cost function will have to be minimized numerically. While the flexibility of the gamma distribution is the

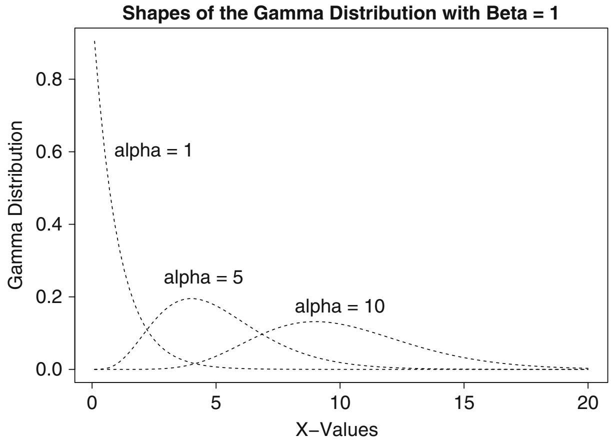


Fig. 1 Gamma distribution with scale parameter $\beta = 1$ and different values for the shape parameter α

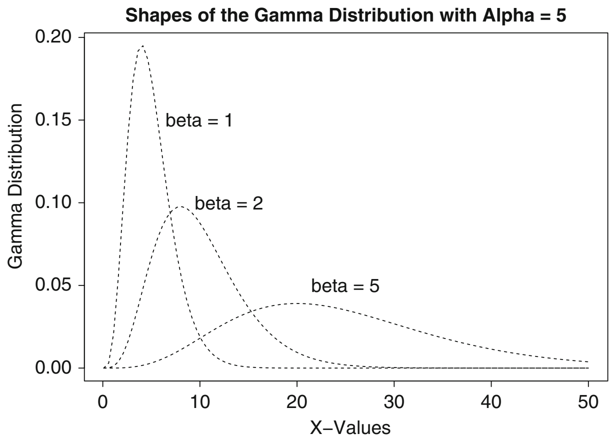


Fig. 2 Gamma distribution with shape parameter $\alpha = 5$ and different values for the scale parameter β

primary source of its attraction, it is also the source of potential intractability. As illustrated in Fig. 1 and Fig. 2, similar gamma distributions can be constructed from different combinations of α and β. By fixing α we still have sufficient flexibility to parameterize a variety of gamma distributions with different shapes.

For the double exponential distribution, we treat the scale parameter $\beta = \beta_0$ as a system parameter and optimize over α. As displayed in Fig. 3, the double exponential distribution can become so peaked that in extreme cases controlling the mean could be like "finding a needle in a haystack." In practice we should expect large values for the system parameter κ_1.

$$\mathbb{E}[T(\alpha)] = \kappa_1|\alpha|^r + \frac{\kappa_2}{(2\beta_0^2)^{s/2}} + \gamma \exp(\phi(a_0 - y_0)) \left(1 - (a_1\phi\beta_0)^2\right)^{-1} \\ - \gamma a_1\phi\alpha - \gamma\left[\phi(a_0 - y_0) - 1\right]. \quad (42)$$

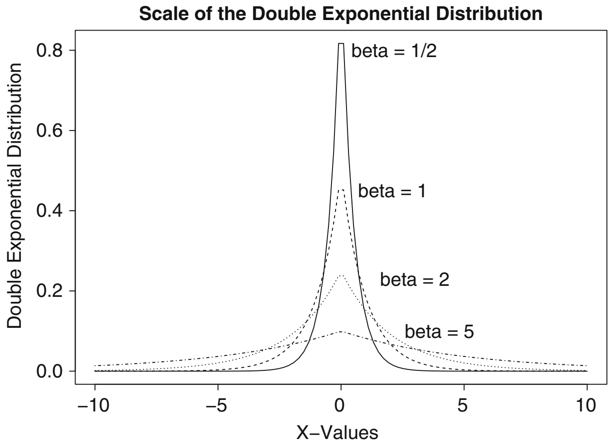


Fig. 3 Double exponential distribution with location parameter $\alpha = 0$ and different values for the scale parameter β

As a function of α, Eq. 42 can be minimized analytically when $r \geq 2$. In this case,

$$\frac{\partial}{\partial \alpha}\mathbb{E}[T(\alpha)] = r\kappa_1\alpha^{r-1} - \gamma a_1 \phi \tag{43}$$

which yields the minimizer

$$\alpha^* = \left(\frac{\gamma a_1 \phi}{r\kappa_1}\right)^{1/(r-1)}. \tag{44}$$

When $a_1\phi > 0$, the neighborhood of α^* is locally quadratic with

$$\frac{\partial^2}{\partial \alpha^2}\mathbb{E}[T(\alpha^*)] = r(r-1)\kappa_1\left(\frac{\gamma a_1 \phi}{r\kappa_1}\right)^{1/(r-1)} > 0. \tag{45}$$

3.1 Numerical examples

We consider two numerical examples similar to the examples of Chang and Hung (2007) and Huang (2001) as illustrations. Let the cost to control the mean and standard deviation of the input quality characteristic be given by $C_X(\mu, \sigma) = 2\mu^2 + 1/\sigma^2$ by setting $\kappa_1 = 2$, $r = 2$, $\kappa_2 = 1$ and $s = 2$. Let the relationship between the input and output quality characteristics be given by the affine function $Y = 50 + 5X$ by setting $a_0 = 50$ and $a_1 = 5$, and set the loss of quality penalty $\gamma = 5$, the LINEX parameter $\phi = 1$, and the target value $y_0 = 100$.

In the case of the gamma input characteristic we will also set the shape parameter $\alpha_0 = 5$. Substituting these system parameters into Eq. 39, we obtain the expected total cost function

$$\mathbb{E}[T(\beta)] = 2|5\beta|^2 + \frac{2}{5\beta^2} + 5\exp(-50)(1-5\beta)^{-5} - 25(5\beta) + 255. \tag{46}$$

Using the BFGS quasi-Newton algorithm for numerical optimization, we obtain a minimizer $\beta^* = 1.252$ with a minimum expected total cost of 177.002. The optimal mean is $\alpha_0\beta^* = 6.260$, and the optimal variance is $\alpha_0(\beta^*)^2 = 7.837$. The objective function in the neighborhood of the minimizer for this example is locally quadratic (see Fig. 4).

Fig. 4 Expected total cost function for the gamma input characteristic in the neighborhood of the minimizer

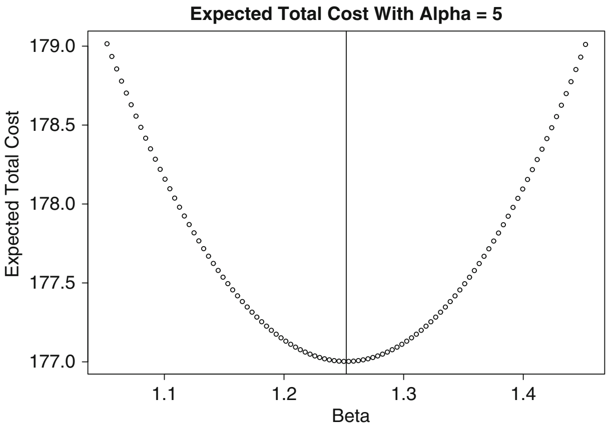
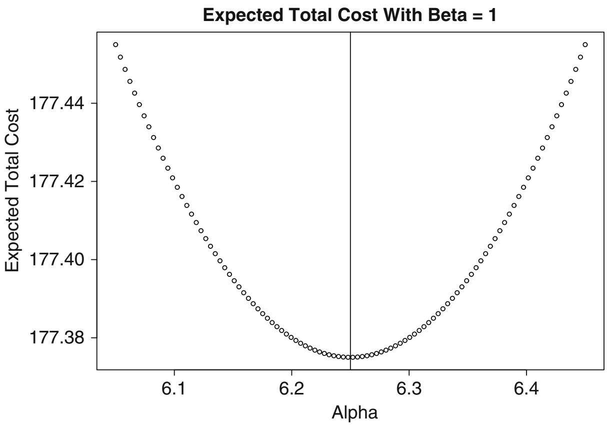


Fig. 5 Expected total cost for the double exponential input characteristic in the neighborhood of the minimizer



For the double exponential input characteristic, we will set the scale parameter $\beta_0 = 1$. Substituting the system parameters into Eq. 40, we obtain the expected total cost function

$$\mathbb{E}[T(\alpha)] = 2|\alpha|^2 + \frac{1}{2} + 5\exp(-50)\left(1 - (5)^2\right)^{-1} - 25\alpha + 255. \tag{47}$$

Using the BFGS quasi-Newton algorithm for numerical optimization, we obtain a minimizer $\alpha^* = 6.250$ with a minimum expected total cost of 177.375. The value for α^* is the optimal mean for the fixed variance $2\beta_0 = 2$, and it is also the minimizer obtained from the analytical solution. Figure 5 displays the locally quadratic behavior of the objective function in the neighborhood of α^*.

4 Conclusions

In this paper we have generalized the work of Chang and Hung (2007) to include the gamma distribution and the double exponential distribution in the Taguchi quality model with the LINEX loss function and the trade-off between quality and cost proposed by Huang (2001) for the case when the input–output transformation $Y = g(X)$ is an affine function. Both the gamma distribution and the double exponential distribution represent important alternative distributions for departures from normality. The family of gamma distributions contains the exponential and chi-squared distributions as specific cases, and it provides a flexible family of distributions which may be used to approximate non-negative and asymmetric random variables such as the log-normal distribution. The double exponential distribution provides an alternative to the two most popular symmetric distributions: the normal distribution and the t-distribution. Since the moment generating function does not exist for the t-distribution, the double exponential distribution is a symmetric heavy tailed distribution which can be used with the LINEX loss function.

In addition to deriving sufficient conditions for the existence of a global minimum for the Taguchi model with LINEX loss function in each case, we have also considered the reduced parameter restricted cases. Due to the large number of parameters from the LINEX loss function and the transformation $Y = g(X)$ and the necessity of numerical solutions, we can only make limited general statements. However, these general statements can be used effectively for the solution of particular problems, in particular the problems which may arise in practice. In the numerical examples we demonstrate that solutions to the trade-off problem between quality and cost can be easily obtained and well-behaved. While the gamma and double exponential distributions are generally more difficult to use than the normal distribution for analytical solutions, they can be numerically solved and offer useful alternatives to accommodate the departures from normality frequently found in practice.

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