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MINIMAX INVARIANT ESTIMATOR OF CONTINUOUS DISTRIBUTION FUNCTION UNDER LINEX LOSS*

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Abstract In this paper we consider the problem of estimation of a continuous distribution function under the LINEX loss function. The best invariant estimator is obtained, and proved to be minimax for any sample size $n \geq 1$.

Key words Invariant estimator, LINEX loss function, non-parametric estimation.

1 Introduction

The best invariant estimator for a continuous distribution function under monotone transformations and the weighted Cramer-von Mises loss function was introduced by Aggarwal[1]. Yu[2] proved that the best invariant estimator is minimax for $n \geq 1$ under the loss

$$L(F, d) = \int |F(t) - d(t)|^r h(F(t))\, dF(t), \tag{1}$$

where $r \geq 1$, h is a nonnegative weight function and d is a nondecreasing function from $\boldsymbol{R}$ into $[0, 1]$. Yu and Phadia[3] proved that the best invariant estimator is minimax for $n \geq 1$ under the Kolmogorov-Smirnov loss function

$$L(F, d) = \sup_t |F(t) - d(t)|. \tag{2}$$

It is interesting to note that virtually all of the aforementioned studies relate only to symmetric loss. Being symmetric, the loss imposes equal penalty on over- and under-estimation of the same magnitude. There are situations where over- and under-estimation can lead to different consequences. For example, when estimating the average life of the components of a spaceship or an aircraft, over-estimation is usually more serious than under-estimation. In fact, Feynman's report[4] suggests that the space shuttle disaster of 1986 was partly the result of the management's over-estimation of the average life of the solid fuel rocket booster. Zellner[5] also suggested that in dam construction, under-estimation of the peak water level is often much more serious than over-estimation. These examples illustrate that in many situations, the symmetric loss function can be unduly restrictive and inappropriate, and suggest that we should consider

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properties of estimators based on an asymmetric loss function instead. In a study of real estate assessment, Varian[6] introduced the following asymmetric linear exponential (LINEX) loss function:

$$L(\theta,\delta) = b(\exp\{a(\delta-\theta)\} - a(\delta-\theta) - 1), \tag{3}$$

where $a \neq 0$ is a shape parameter and $b > 0$ is a factor of proportionality. The LINEX loss reduces to quadratic loss for small values of a. If a is positive (negative), then over (under)-estimation is considered to be more serious than under (over)-estimation of the same magnitude, and vice versa. Numerous authors, such as Zellner[5], Parsian[7], Takagi[8], Cain and Janssen[9], Ohtani[10], Zou[11], Wan[12], Wan and Kurumai[13] and Wana, Zou and Lee[14] have considered the LINEX loss in various parametric problems. But all of the aforesaid studies only used the LINEX loss to estimation of parameters.

In this paper, we want to consider the LINEX loss as the measure of the distance between a distribution function and its estimator. In order to use the loss to estimation of a distribution function, we need to modify the loss (3). We replace θ and δ with $F(t)$ and d, respectively. Then the loss (3) changes as

$$L'(F(t),d(t)) = b(\exp\{a(d(t)-F(t))\} - a(d(t)-F(t)) - 1). \tag{4}$$

Since (4) relates to variable t, we consider the mean of L' about a finite measure $W(t)$:

$$L''(F,d) = b\int(\exp\{a(d(t)-F(t))\} - a(d(t)-F(t)) - 1)\,dW(t). \tag{5}$$

Considering the invariant problem, we let $W(t) = F(t)$. Then, we get the invariant nonparametric LINEX loss

$$L(F(t),d(t)) = b\int(\exp\{a(d(t)-F(t))\} - a(d(t)-F(t)) - 1)\,dF(t). \tag{6}$$

In Section 2 we obtain the best invariant estimator of F under the loss (6). In Section 3 we prove that the best invariant estimator is minimax under the loss (6) for any sample size $n \geq 1$.

2 The Best Invariant Estimator

Let $x_1, x_2, \cdots, x_n$ be a random sample of size n from an unknown continuous type distribution function F. Suppose $X = (x_{(1)}, x_{(2)}, \cdots, x_{(n)})$ is the vector of order statistics of $x_1, x_2, \cdots, x_n$, $x_{(0)} = -\infty$, and $x_{(n+1)} = +\infty$. The action space and parameter space are given by

$$\mathcal{A} = \{d:\ d \text{ is a nondecreasing function from } \boldsymbol{R} \text{ into } [0,1]\} \tag{7}$$

and

$$\Theta_c = \{F(t):\ F \text{ is a continuous c.d.f on } \boldsymbol{R}\}, \tag{8}$$

respectively. All estimators which we consider are functions of the order statistics X, since they form an essentially complete class[15]. For convenience, we write $d = d(t) = d(X,t) = d(x_{(1)}, x_{(2)}, \cdots, x_{(n)}, t)$. It is well known[1,16] that the estimator which is invariant under the group of transformations

$$\mathcal{G} = \big\{g_\varphi:\ g_\varphi(x_1, x_2, \cdots, x_n) = (\varphi(x_1), \varphi(x_2), \cdots, \varphi(x_n)),\ \varphi \text{ is a continuous and strictly increasing function from } \boldsymbol{R} \text{ to } \boldsymbol{R}\big\}. \tag{9}$$

is of the form

$$d(t, X) = \sum_{k=0}^{n} u_k 1(x_{(k)} \le t < x_{(k+1)}), \tag{10}$$

where $1(A)$ is the indicator function of the set A and the u_k is constant with $0 \le u_k \le u_{k+1} \le 1$. For convenience, we denote the set of all invariant estimator of form (10) by $\mathcal{U}$. Thus, to find the best invariant estimator in $\mathcal{U}$, we have to determine the u_k suitably. The best invariant estimator of F under the loss (6) is obtained as follows.

For any invariant estimator $d \in \mathcal{U}$, we have

$$\begin{aligned} L(F,\, d) &= b \int (\exp\{a(d(t) - F(t))\} - a(d(t) - F(t)) - 1)\, dF(t) \\ &= b \sum_{i=0}^{n} \int_{x_{(i)}}^{x_{(i+1)}} (\exp\{a(d(t) - F(t))\} - a(d(t) - F(t)) - 1)\, dF(t) \\ &= b \sum_{i=0}^{n} \int_{F(x_{(i)})}^{F(x_{(i+1)})} (\exp\{a(u_i - t)\} - a(u_i - t) - 1)\, dt \\ &= b \sum_{i=0}^{n} L_i(F,\, d), \end{aligned} \tag{11}$$

where

$$L_i(F,\, d) = \int_{F(x_{(i)})}^{F(x_{(i+1)})} (\exp\{a(u_i - t)\} - a(u_i - t) - 1)\, dt.$$

Let $Z_i = F(x_i)$. Then, we have

$$\begin{aligned} R(F, d) = E[L(F, d)] &= b \sum_{i=0}^{n} E\left(L_i(F, d)\right) \\ &= b \sum_{i=0}^{n} E \int_{Z_i}^{Z_{i+1}} (\exp\{a(u_i - t)\} - a(u_i - t) - 1) dt \\ &= b \sum_{i=0}^{n} \int_0^1 \int_0^{Z_{i+1}} \int_{Z_i}^{Z_{i+1}} (\exp\{a(u_i - t)\} - a(u_i - t) - 1)\, dt dF_{z_i z_{i+1}}(z_i, z_{i+1}) \\ &= b \sum_{i=0}^{n} \int_0^1 \int_t^1 \int_0^t (\exp\{a(u_i - t)\} - a(u_i - t) - 1)\, dF_{z_i z_{i+1}}(z_i, z_{i+1}) dt \\ &= b \sum_{i=0}^{n} \int_0^1 (\exp\{a(u_i - t)\} - a(u_i - t) - 1) \int_t^1 \int_0^t dF_{z_i z_{i+1}}(z_i, z_{i+1}) dt \\ &= b \sum_{i=0}^{n} \int_0^1 (\exp\{a(u_i - t)\} - a(u_i - t) - 1) \binom{n}{i} t^i (1-t)^{n-i} dt \\ &= b \sum_{i=0}^{n} R_i(F, d), \end{aligned} \tag{12}$$

where

$$R_i(F, d) = \int_0^1 \binom{n}{i} (\exp\{a(u_i - t)\} - a(u_i - t) - 1) t^i (1-t)^{n-i} dt.$$

Noticing the integrated function in $R_i(F, d)$, as the function of (t, u_i), is continuous, and its first and second order derivatives with respect to u_i, as the function of (t, u_i), are also continuous.

So we can alternate the order of differential and integral in solving the derivatives of $R_i(F,d)$ with respect to u_i and can get the following first and second order derivatives

$$\frac{\partial}{\partial u_i}R_i(F,d)=\int_0^1\binom{n}{i}(a\exp\{a(u_i-t)\}-a)t^i(1-t)^{n-i}dt, \tag{13}$$

$$\frac{\partial^2}{\partial u_i^2}R_i(F,d)=\int_0^1\binom{n}{i}a^2\exp\{a(u_i-t)\}t^i(1-t)^{n-i}dt. \tag{14}$$

It is clear from (14) that $\frac{\partial^2}{\partial u_i^2}R_i(F,\,d)>0$ for any $i=0,1,\cdots,n$. Thus $R_i(F,d)$, and hence $R(F,d)$, is minimized by setting $\frac{\partial}{\partial u_i}R_i(F,\,d)=0$. Solving the equation $\frac{\partial}{\partial u_i}R_i(F,\,d)=0$, we get

$$\exp\{au_i\}=\frac{\int_0^1 t^i(1-t)^{n-i}dt}{\int_0^1\frac{1}{\exp\{at\}}t^i(1-t)^{n-i}dt}.$$

Then

$$u_i=\frac{1}{a}\ln\frac{\int_0^1 t^i(1-t)^{n-i}dt}{\int_0^1\frac{1}{\exp\{at\}}t^i(1-t)^{n-i}dt}.$$

It will be proved that $0\le u_i\le u_{i+1}\le 1$ in appendix. So we obtain the best invariant estimator

$$\phi(t,X)=\sum_{k=0}^{n}u_k1(x_{(k)}\le t<x_{(k+1)}), \tag{15}$$

where

$$u_i=\frac{1}{a}\ln\frac{\int_0^1 t^i(1-t)^{n-i}dt}{\int_0^1\frac{1}{\exp\{at\}}t^i(1-t)^{n-i}dt}.$$

From (12), we have the following remark.

Remark 2.1 The risk $R(F,d)$ of an invariant estimator d is a constant.

3 Minimaxity of the Best Invariant Estimator

In this section, it is proved that the best invariant estimator ϕ is minimax for all sample sizes $n\ge 1$ under the losses (6). Given a distribution function F, let P be the measure induced by F, that is, $P\{(a,b)\}=F(b)-F(a)$; Let P^m denote the product measure $P\times\cdots\times P$ with m factors, $m=1,2,\cdots$. The proof of our main result depends on the following lemma.

Lemma 3.1[15,Theorem 4] *Suppose that $d=d(X,t)$ is a nonrandomized estimator with the finite risk and a measurable function of the order statistics $X=(x_{(1)},x_{(2)},\cdots,x_{(n)})$. For any $\varepsilon,\delta>0$, there exists a uniform distribution P_0 on a Lebesgure-measurable subset $J\subset R$ and an invariant estimator d_1 such that*

$$P_0^{n+1}\{(X,t):|d(X,t)-d_1(X,t)|\ge\varepsilon\}\le\delta.$$

Remark 3.1 It is obviously that

$$0<L(F,d)=b\int(\exp\{a(d(t)-F(t))\}-a(d(t)-F(t))-1)\,dF(t)<+\infty,$$

for each $d\in\mathcal{A}$ and $F\in\Theta_c$, and hence $R(F,d)$ is always finite for any $d\in\mathcal{A}$. So the qualification that d's risk is finite in Lemma 3.1 is always right.

The following lemma is needed in later proof.

Lemma 3.2 *Consider the function* $\exp\{a(x-s)\}$, *where* $a \neq 0$, $0 < s < 1$ *are both constants. Then, for* $0 \leq x_1 < x_2 \leq 1$, *we have*

$$|\exp\{a(x_1-s)\} - \exp\{a(x_2-s)\}| \leq |a||x_1-x_2| \exp|a|. \tag{16}$$

Proof Obviously, the function $\exp\{a(x-s)\}$ is continuous, so there exits $\xi \in (x_1, x_2)$ such that

$$\begin{aligned}(\exp\{a(x_1-s)\} - \exp\{a(x_2-s)\}) &= \left\{\frac{\partial}{\partial x}\exp\{a(x-s)\}\right\}\bigg|_{x=\xi}(x_1-x_2)\\ &= a(x_1-x_2)\exp\{a(\xi-s)\}.\end{aligned} \tag{17}$$

On the other hand, we have

$$\begin{cases}\exp\{a(\xi-s)\} \leq \exp\{a(1-s)\} \leq \exp\{a\}, & \text{for} \quad a>0,\\ \exp\{a(\xi-s)\} \leq \exp\{-as\} \leq \exp\{-a\}, & \text{for} \quad a<0.\end{cases} \tag{18}$$

Substituting from (18) in (17) we can obtain

$$|\exp\{a(x_1-s)\} - \exp\{a(x_2-s)\}| \leq |a||x_1-x_2| \exp|a|.$$

This completes the proof of Lemma 3.2. ∎

Now we state another lemma which leads to the proof of minimaxity.

Lemma 3.3 *Suppose that the sample size* $n \geq 1$ *and* $\varepsilon > 0$. *Consider the loss* (6). *Then for a given* $d \in \mathcal{A}$, *there exist* $F \in \Theta_c$ *and invariant estimator* $d_0 \in \mathcal{U}$ *such that* $|R(F,d) - R(F,d_0)| \leq \varepsilon$.

Proof By Lemma 3.1 there exist $P_0 \in \Theta_c$ and invariant estimator d_0 such that $P_0^{n+1}\{(X,t) : |d(X,t) - d_0(X,t)| \geq \varepsilon s\} \leq \varepsilon s$, where $s = (2|a|(\exp(|a|)+1))^{-1}$. Let $B = \{(X,t) : |d(t) - d_0(t)| \geq \varepsilon s\}$ and F denote the distribution function corresponding to P_0, then

$$\begin{aligned}&|R(F,\, d) - R(F,\, d_0)|\\ =&|E[L(F,d)] - E[L(F,d_0)]| \leq bE\{|L(F,d) - L(F,d_0)|\}\\ \leq& bE\int_0^1 |\exp\{a(d-F)\} - \exp\{a(d_0-F)\} + a(d_0-d)|\, dF\\ \leq& bE\int_0^1 |\exp\{a(d-F)\} - \exp\{a(d_0-F)\}| + |a(d_0-d)|\, dF\\ \leq& bE\int_0^1 \{|a|\exp(|a|)|d_0-d| + |a||d_0-d|\}\, dF \quad \text{(by Lemma 3.2)}\\ \leq& bE\int_0^1 \{|a|(\exp(|a|)+1)|d-d_0|\}\, dF\\ =&b|a|(\exp(|a|)+1)\int |d-d_0| dF(t) dF(X)\\ =&b|a|(\exp(|a|)+1)\left\{\int_B |d-d_0| dF(t) dF(X) + \int_{B^-} |d-d_0| dF(t) dF(X)\right\}\\ \leq& b|a|(\exp(|a|)+1)(P_0^{n+1}(B) + \varepsilon s) \qquad \text{(by } |\mathrm{d}-\mathrm{d}_0| \leq 1 \text{ and Lemma 3.1)}\\ =&b|a|(\exp(|a|)+1)(\varepsilon s + \varepsilon s) \leq \varepsilon.\end{aligned}$$

This completes the proof. ∎

Now we can prove the minimaxity of the best invariant estimator (15).

Theorem 3.1 *Under the loss* (6)*, the best invariant estimator* ϕ *of* $F \in \Theta_c$ *is minimax for* $n \geq 1$.

Proof From Lemma 3.3, for any $d \in \mathcal{A}$, $F \in \Theta_c$, and $\varepsilon > 0$, there exits a $F_0 \in \Theta_c$ and an invariant estimator $d_0 \in \mathcal{U}$, such that $|R(F_0, d) - R(F_0, d_0)| \leq \varepsilon$. Especially, we have

$$R(F_0, d_0) \leq R(F_0, d) + \varepsilon \leq \sup_{F \in \Theta_c} R(F, d) + \varepsilon.$$

From the best invariant estimator property of ϕ we have

$$R(F_0, \phi) \leq R(F_0, d_0) \leq R(F_0, d) + \varepsilon \leq \sup_{F \in \Theta_c} R(F, d) + \varepsilon. \tag{19}$$

Furthermore, we know that ϕ has constant risk in Θ_c from Remark 2.1. Then we have

$$\sup_{F \in \Theta_c} R(F, \phi) = R(F_0, \phi) \leq \sup_{F \in \Theta_c} R(F, d) + \varepsilon. \tag{20}$$

Since d and ε are arbitrary, we can say

$$\sup_{F \in \Theta_c} R(F, \phi) = \inf_{d \in \mathcal{A}} \sup_{F \in \Theta_c} R(F, d).$$

That is, the best invariant estimator ϕ of F is minimax. ∎

4 Appendix

Next the proof of $0 \leq u_i \leq u_{i+1} \leq 1$ in (15) is given.

We prove $0 \leq u_i \leq 1$ firstly. Note that

$$u_i = \frac{1}{a} \ln \frac{\int_0^1 t^i (1-t)^{n-i} dt}{\int_0^1 \frac{1}{\exp\{at\}} t^i (1-t)^{n-i} dt}.$$

Then $0 \leq u_i \leq 1$ is equivalent to

$$\begin{cases} 1 \leq \dfrac{\int_0^1 t^i (1-t)^{n-i} dt}{\int_0^1 \frac{1}{\exp\{at\}} t^i (1-t)^{n-i} dt} \leq e^a, & \text{if } \ a > 0, \\ e^a \leq \dfrac{\int_0^1 t^i (1-t)^{n-i} dt}{\int_0^1 \frac{1}{\exp\{at\}} t^i (1-t)^{n-i} dt} \leq 1, & \text{if } \ a < 0. \end{cases} \tag{A1}$$

Moreover, (A1) is equivalent to

$$\begin{cases} \displaystyle\int_0^1 \frac{1}{\exp\{at\}} t^i (1-t)^{n-i} dt \leq \int_0^1 t^i (1-t)^{n-i} dt \leq \int_0^1 \exp\{a - at\} t^i (1-t)^{n-i} dt, & \text{if } a > 0, \\ \displaystyle\int_0^1 \exp\{a - at\} t^i (1-t)^{n-i} dt \leq \int_0^1 t^i (1-t)^{n-i} dt \leq \int_0^1 \frac{1}{\exp\{at\}} t^i (1-t)^{n-i} dt, & \text{if } a < 0. \end{cases} \tag{A2}$$

Note that the fact of

$$\begin{cases} \exp\{at\} \geq 1, & \text{if } \ a > 0, \ \ t \in [0,1], \\ 0 < \exp\{at\} \leq 1, & \text{if } \ a < 0, \ \ t \in [0,1], \end{cases} \tag{A3}$$

implies (A2). This completes the proof of $0 \le u_i \le 1$.

Now let us show the monotonicity of u_i with respect to the index i. Let

$$Z_i = \frac{\int_0^1 \frac{1}{\exp\{at\}} t^i (1-t)^{n-i} dt}{\int_0^1 t^i (1-t)^{n-i} dt}.$$

Then $u_i = \frac{1}{a} \ln\{\frac{1}{Z_i}\}$. Note that

$$\begin{aligned} Z_i &= \frac{\int_0^1 \sum_{k=0}^{+\infty} (-1)^k \frac{(at)^k}{k!} t^i (1-t)^{n-i} dt}{\int_0^1 t^i (1-t)^{n-i} dt} = \sum_{k=0}^{+\infty} (-1)^k \frac{a^k}{k!} \frac{\int_0^1 t^{i+k} (1-t)^{n-i} dt}{\int_0^1 t^i (1-t)^{n-i} dt} \\ &= \sum_{k=1}^{+\infty} (-1)^k \frac{a^k}{k!} \frac{\Gamma(i+k+1)\Gamma(n-i+1)}{\Gamma(n+k+2)} \left(\frac{\Gamma(i+1)\Gamma(n-i+1)}{\Gamma(n+2)} \right)^{-1} + 1 \\ &= \sum_{k=1}^{+\infty} (-1)^k \frac{a^k}{k!} \frac{(k+i)\cdot(k+i-1)\cdots(i+1)}{(k+1+n)\cdot(k+n)\cdots(n+2)} + 1. \end{aligned} \tag{A4}$$

Then

$$\begin{aligned} &Z_{i+1} - Z_i \\ &= \left\{ \sum_{k=1}^{+\infty} (-1)^k \frac{a^k}{k!} \frac{(k+i)\cdot(k+i-1)\cdots(i+1)}{(k+1+n)\cdot(k+n)\cdots(n+2)} + 1 \right\} \\ &\quad - \left\{ \sum_{k=1}^{+\infty} (-1)^k \frac{a^k}{k!} \frac{(k+i+1)\cdot(k+i)\cdots(i+2)}{(k+1+n)\cdot(k+n)\cdots(n+2)} + 1 \right\} \\ &= \sum_{k=2}^{+\infty} (-1)^k \frac{a^k}{k!} \frac{(k+i)\cdots(i+2)}{(k+1+n)\cdot(k+n)\cdots(n+2)} \cdot k - \frac{a}{n+2} \\ &= (-a) \sum_{k=2}^{+\infty} (-1)^{k-1} \frac{a^{k-1}}{(k-1)!} \frac{(k+i)\cdots(i+2)}{(k+1+n)\cdot(k+n)\cdots(n+2)} - \frac{a}{n+2} \\ &= (-a) \sum_{k=1}^{+\infty} (-1)^k \frac{a^k}{k!} \frac{(k+1+i)\cdots(i+2)}{(k+2+n)\cdot(k+n)\cdots(n+2)} - \frac{a}{n+2} \\ &= \frac{(-a)}{n+2} \left\{ \sum_{k=1}^{+\infty} (-1)^k \frac{a^k}{k!} \frac{(k+1+i)\cdots(i+2)}{(k+2+n)\cdot(k+n)\cdots(n+3)} + 1 \right\} \\ &= \frac{(-a)}{n+2} \left\{ \sum_{k=1}^{+\infty} (-1)^k \frac{a^k}{k!} \frac{\Gamma(i+k+2)\Gamma(n-i+1)}{\Gamma(n+k+3)} \left(\frac{\Gamma(i+2)\Gamma(n-i+1)}{\Gamma(n+3)} \right)^{-1} + 1 \right\} \\ &= \frac{(-a)}{n+2} \sum_{k=0}^{+\infty} (-1)^k \frac{a^k}{k!} \frac{\int_0^1 t^{i+k+1} (1-t)^{n-i} dt}{\int_0^1 t^{i+1} (1-t)^{n-i} dt} \\ &= \frac{(-a)}{n+2} \frac{\int_0^1 \sum_{k=0}^{+\infty} (-1)^k \frac{(at)^k}{k!} t^{i+1} (1-t)^{n-i} dt}{\int_0^1 t^{i+1} (1-t)^{n-i} dt} \\ &= \frac{(-a)}{n+2} \frac{\int_0^1 \exp\{-at\} t^{i+1} (1-t)^{n-i} dt}{\int_0^1 t^{i+1} (1-t)^{n-i} dt}. \end{aligned} \tag{A5}$$

Since

$$\frac{\int_0^1 \exp\{-at\}t^{i+1}(1-t)^{n-i}dt}{\int_0^1 t^{i+1}(1-t)^{n-i}dt} > 0,$$

from (A5), we have

$$\begin{cases} Z_{i+1} \le Z_i, & \text{if} \quad a > 0, \\ Z_{i+1} \ge Z_i, & \text{if} \quad a < 0. \end{cases} \tag{A6}$$

Hence we obtain

$$u_i = \frac{1}{a}\ln\left\{\frac{1}{Z_i}\right\} \le u_{i+1} = \frac{1}{a}\ln\left\{\frac{1}{Z_{i+1}}\right\}, \qquad \text{for any } a \ne 0. \qquad \blacksquare$$

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