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Convergence rate of empirical Bayes estimation for two-dimensional truncation parameters under linex loss

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Abstract

This paper deals with the empirical Bayes (EB) estimation for two-dimensional truncation parameters under linex loss. By using kernel estimation of density function, we construct the EB estimation of parameters. Under some suitable conditions, we prove that the constructed EB estimations is asymptotically optimal and we obtain its convergence rate.

Keywords: Linex loss; Kernel function; Empirical Bayes estimation; Convergence rate

1. Introduction

In many estimation problems, when the positive and negative estimation errors of the same magnitude have different consequences, the quadratic loss function may be inappropriate for deriving and comparing the performance of

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various estimators. To overcome the deficiency of the quadratic loss function, Varian [6] introduced an asymmetric loss function, namely linex loss function, to study the real estate assessment. More properties of the loss function have been investigated by Zellner [7]. Basu and Ebrahimi used linex loss function in lifetime testing and reliability estimation [1,2], and Huang and Liang [3] studied the empirical Bayes estimation of the truncation parameter under linex loss [5]. But the EB estimation for the parameters two-dimensional one-side truncated distribution was not addressed.

Consider the two-dimensional truncated distribution family with probability density function (pdf) of the following form

$$f(x_1,x_2|\theta) = u(x_1,x_2)\phi(\theta_1,\theta_2)I_{(D)}(x_1,x_2) \tag{1.1}$$

where $D = (\theta_1,b_1) \times (\theta_2,b_2)$, $u(x_1,x_2)$ is a nonnegative and integrable function, and $\theta = (\theta_1,\theta_2)$ is the parameter of our interest, $\theta_i > a_i \geqslant 0$, $b_i \leqslant +\infty$, $i = 1,2$, and $\phi(\theta_1,\theta_2) = [\int_{\theta_1}^{b_1}\int_{\theta_2}^{b_2} u(x_1,x_2)\,\mathrm{d}x_1\,\mathrm{d}x_2]^{-1}$.

Let the loss function (Linex loss) be $L(\theta_1,\theta_2;d_1,d_2) = \sum_{i=1}^{2}\{\exp\{c_i(d_i - \theta_i)\} - c_i(d_i-\theta_i) - 1\}$ where d_i denotes an estimation of θ_i $i = 1,2$, and $c_i \in R$, $c_i \neq 0$ is a constant. In this paper, we consider only $c_i > 0$. Discussion of the other case where $c_i < 0$, is similar and therefore omitted here.

The paper is organized as follows. In Section 2, the EB estimator for the truncation parameters are derived under Linex loss, and a theorem about convergence rate is given. In Section 3, some lemmas are given, and the proof of theorem is completed. At last, we give an example satisfying the conditions of the theorem.

2. Empirical Bayes estimation

Let $\Omega = (a_1,b_1) \times (a_2,b_2)$ be the parameter space, and the prior distribution of (θ_1,θ_2) be $G(\theta_1,\theta_2)$ with pdf of $g(\theta_1,\theta_2)$. Then the marginal distribution of $X = (X_1,X_2)$ is

$$\begin{aligned} f(x_1,x_2) &= \int\int_\Omega f(x_1,x_2|\theta)g(\theta_1,\theta_2)\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2 \\ &= \int_{a_1}^{x_1}\int_{a_2}^{x_2} u(x_1,x_2)\phi(\theta_1,\theta_2)g(\theta_1,\theta_2)\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2 = u(x_1,x_2)v(x_1,x_2) \end{aligned}$$

where $v(x_1,x_2) = \int_{a_1}^{x_1}\int_{a_2}^{x_2}\phi(\theta_1,\theta_2)g(\theta_1,\theta_2)\,\mathrm{d}\theta_1\,\mathrm{d}\theta_2$.

If prior distribution $g(\theta_1,\theta_2)$ is given and given $X = x$, then the Bayes estimator of θ_i is $\phi_{iG}(x)$ which minimizes $\int_\Omega\{\exp[c_i(d_i-\theta_i)] - c_i(d_i-\theta_i) - 1\}\,\mathrm{d}G(\theta|x)$.

Some straightforward computation yields

$$\phi_{iG}(x) = -(c_i)^{-1}\ln E(\mathrm{e}^{-c_i x_i}|x) = (c_i)^{-1}\ln \tau_{iG}(x), \quad \text{with}$$
$$\tau_{iG}(x) = [E(\mathrm{e}^{-c_i\theta_i}|x)]^{-1} \quad i = 1,2$$

Assume that $\lim_{x_2\to a_2}\frac{\mathrm{d}}{\mathrm{d}x_1}v(x_1,x_2) = 0$, $\lim_{x_1\to a_1}\frac{\mathrm{d}}{\mathrm{d}x_2}v(x_1,x_2) = 0$, for any given x_1 or x_2.

Then $E(\mathrm{e}^{-c_i\theta_i}|x) = \frac{\mathrm{e}^{-c_i x_i}f(x_1,x_2)+D_{iG}(x_1,x_2)}{f(x_1,x_2)}$, $i = 1,2$ where $D_{1G}(x_1,x_2) = u(x_1,x_2)\times\int_{a_1}^{x_1} c_1\mathrm{e}^{-c_1\theta_1}\frac{f(\theta_1,x_2)}{u(\theta_1,x_2)}\,\mathrm{d}\theta_1$

$$D_{2G}(x_1,x_2) = u(x_1,x_2)\int_{a_2}^{x_2} c_2\mathrm{e}^{-c_2\theta_2}\frac{f(x_1,\theta_2)}{u(x_1,\theta_2)}\,\mathrm{d}\theta_2$$

The Bayes risk, attained from the Bayes estimation $\phi_G(x_1,x_2) = (\phi_{1G}(x_1,x_2), \phi_{2G}(x_1,x_2))$ of θ is

$$\begin{aligned} R_G &= \sum_{i=1,2}^{2} E_{(x,\theta)}\{\exp[c_i(\phi_{iG}(x)-\theta_i)] - c_i(\phi_{iG}(x)-\theta_i) - 1\} \\ &= R_{1G} + R_{2G} \end{aligned} \tag{2.1}$$

where $R_{iG} = E_{(x,\theta)}\{\exp[c_i(\phi_{iG}(x)-\theta_i)] - c_i(\phi_{iG}(x)-\theta_i) - 1\}$, for $i = 1,2$. $E_{(x,\theta)}$ denotes the expectation with respect to the joint distribution of (X,θ). We define that $x^{(i)} = (x_{i1},x_{i2})$, $x = (x_1,x_2)$, $\theta^{(i)} = (\theta_{i1},\theta_{i2})$, $i = 1,2,\ldots,n$. Let $C_{k\alpha}$ be a family of probability densities on Ω, which have the k order continuous mixed partial derivatives whose absolute value is bounded by α. Usually the prior distribution $G(\theta_1,\theta_2)$ is unknown, and the Bayes estimator $\phi_G(x)$ cannot be applied. In this paper, the empirical Bayes approach is introduced to handle the uncertainty of G.

Suppose that $(X^{(1)},\theta^{(1)}),\ldots,(X^{(n)},\theta^{(n)})$ and (X,θ) are independent identical distribution random samples, where $(X^{(1)},X^{(2)},\ldots,X^{(n)})$ denote the past samples, and X is the present sample, $\theta^{(i)}$ $(i = 1,2,\ldots,n)$ and θ have the same distribution $G(\theta_1,\theta_2)$. In order to estimate $f(x)$, we employ the kernel function $K(y)$, which satisfies the following conditions:

(1) $K(y) = 0$, if $y \notin (0,1)$
(2) $|K(y)| < M$, for all $y \in R$, and M is a positive constant
(3) $\int y^t K(y)\,\mathrm{d}y = \begin{cases} 1, & t = 0 \\ 0, & t = 1,2,\ldots,s-1 \end{cases}$ where $s > 1$ is a natural number.

We define the estimator of $f(x_1,x_2)$ by

$$f_n(x_1,x_2) = \frac{1}{nh_n^2}\sum_{i=1}^{n} k\left(\frac{x_{i1}-x_1}{h_n}\right)k\left(\frac{x_{i2}-x_2}{h_n}\right),$$

where $0 < h_n \to 0$, as $n \to \infty$.

Let

$$D_{1n}(x_1,x_2) = u(x_1,x_2)\int_{a_1}^{x_1} \frac{c_1 e^{-c_1\theta_1} f_n(\theta_1,x_2)}{u(\theta_1,x_2)}\,d\theta_1$$

$$D_{2n}(x_1,x_2) = u(x_1,x_2)\int_{a_2}^{x_2} \frac{c_2 e^{-c_2\theta_2} f_n(x_1,\theta_2)}{u(x_1,\theta_2)}\,d\theta_2$$

$$\tau_{\text{in}}(x_1,x_2) = \begin{cases} \frac{f_n(x_1,x_2)}{e^{-c_i\theta_i} f_n(x_1,x_2)+D_{\text{in}}(x_1,x_2)}, & 1 < \frac{f_n(x_1,x_2)}{e^{-c_i\theta_i} f_n(x_1,x_2)+D_{\text{in}}(x_1,x_2)} < n^{v_i} \\ 1, & \text{otherwise} \end{cases} \tag{2.2}$$

$i = 1,2$, where v_i is an unknown constant. $\phi_{\text{in}}(x_1,x_2) = c_i^{-1}\ln\tau_{\text{in}}(x_1,x_2)$, $i = 1,2$.

Define $\phi_n(x_1,x_2) = (\phi_{1n}(x_1,x_2), \phi_{2n}(x_1,x_2))$ as the EB estimator of $\theta = (\theta_1,\theta_2)$. The overall Bayes risk of $\phi_n(x_1,x_2)$ is

$$R_n = E_*\sum_{i=1}^{2}\{\exp[c_i(\phi_{\text{in}}(x_1,x_2)-\theta_i)] - c_i(\phi_{\text{in}}(x_1,x_2)-\theta_i) - 1\} = R_{1n} + R_{2n}$$

where $R_{\text{in}} = E_*\{\exp[c_i(\phi_{\text{in}}(x_1,x_2)-\theta_i)] - c_i(\phi_{\text{in}}(x_1,x_2)-\theta_i) - 1\}$, $i = 1,2$.

Throughout this paper, E_* and E stands respectively for the expectation with respect to the joint distribution of $(X^{(1)},\dots,X^{(n)},(X,\theta))$ and $(X^{(1)},\dots,X^{(n)})$. The expectation $E_{(y)}$ is taken with respect to Y. M_1, M_2 and M can denote different positive constants in different cases even in the same expression. Denote

$$\Lambda_1(x_1,x_2) = \sup_{0\leqslant u_i\leqslant 1} f(x_1+u_1, x_2+u_2),$$

$$\Lambda_2(x_1,x_2) = \sup_{\substack{l_1+l_2=s\\ 0\leqslant u_i\leqslant 1}} \left|\frac{\partial^s f(t_1,t_2)}{\partial t_1^{l_1}\partial t_2^{l_2}}\right|_{t=x+u}, \qquad \text{for } i = 1,2$$

where $t = (t_1,t_2), u = (u_1,u_2)$.

The main results of this paper can be expressed in the following theorem.

Theorem. *Let $0 < \lambda < 1$ be a constant and $\mu > 2$ is a given natural number. Assume that the following conditions hold*:

(1) $$E_{(x)}\left|\tau_{iG}(x_1,x_2)\right|^{\mu} < \infty, \quad \textit{for } i = 1,2 \tag{2.3}$$

(2) *$u(x_1,x_2)$ is a bounded and continuous function, $f(x_1,x_2)$ which has k order partial derivatives.*

(3)
$$\lim_{x_2\to a_2}\frac{\mathrm{d}}{\mathrm{d}x_1}v(x_1,x_2)=0,\quad \lim_{x_1\to a_1}\frac{\mathrm{d}}{\mathrm{d}x_2}v(x_1,x_2)=0$$

(4)
$$E_{(x)}\left\{(\mathrm{e}^{-c_1x_1}f(x)+D_{1G}(x))^{-2\lambda}(\Lambda_2^{2\lambda}(x)+\Lambda_1^{\lambda}(x)\right.$$
$$\left.+\left(\int_{a_1}^{x_1}\frac{\Lambda_2(\theta_1,x_2)+\Lambda_1^{1/2}(\theta_1,x_2)}{u(\theta_1,x_2)}\,\mathrm{d}\theta_1)^{\lambda}\right)\right\}<\infty$$

$$E_{(x)}\left\{(\mathrm{e}^{-c_2x_2}f(x)+D_{2G}(x))^{-2\lambda}(\Lambda_2^{2\lambda}(x)+\Lambda_1^{\lambda}(x)\right.$$
$$\left.+\left(\int_{a_2}^{x_2}\frac{\Lambda_2(x_1,\theta_2)+\Lambda_1^{1/2}(x_1,\theta_2)}{u(x_1,\theta_2)}\,\mathrm{d}\theta_2)^{\lambda}\right)\right\}<\infty$$

Then when $h_n=n^{\frac{-1}{2(s+1)}}$, $v_1=v_2=\frac{\lambda s}{\mu(s+1)}$, *we have* $R_n-R_G=O(n^{-q})$, $q=\frac{\lambda s(\mu-2)}{\mu(s+1)}$.

3. Lemmas and proof of theorem

Lemma 1 (See [3]). *For any constant* $a,b\geqslant 0$, *we have* $\mathrm{e}^{a-b}-(a-b)-1\leqslant(\mathrm{e}^a-\mathrm{e}^b)^2$.

Lemma 2. *Let* Y *and* Y' *be random variables*, y *and* y' *be real numbers*, *and* $L>0$ *and* $0<r\leqslant 2$ *be respectively real numbers. Then*

$$E\left|\left[\frac{Y'}{Y}-\frac{y'}{y}\right]_L\right|^r\leqslant 2|y|^{-r}\left\{E|Y'-y'|^r+\left(\left|\frac{y'}{y}\right|+L\right)^r E|Y-y|^r\right\}$$

Proof. See [8], where $[b]_L=\begin{cases}b, & |b|\leqslant L,\\ 0, & |b|>L.\end{cases}$ □

Lemma 3. *If* $f(x_1,x_2)$ *has* s *order partial derivatives* $0<\lambda_1\leqslant 1$ *then when* $h_n=n^{\frac{-1}{2s+2}}$

$$E|f_n(x_1,x_2)-f(x_1,x_2)|^{2\lambda_1}<Mn^{-\frac{\lambda_1 s}{s+1}}[\Lambda_2^{2\lambda_1}(x)+\Lambda_1^{\lambda_1}(x)]$$

Proof. See [4]. □

Lemma 4. *If* $R_G<\infty$, *then* $R_n-R_G=\sum_{i=1}^{2}E_*\{\exp[c_i(\phi_{\mathrm{in}}(x)-\phi_{iG}(x))]-c_i[\phi_{\mathrm{in}}(x)-\phi_{iG}(x)]-1\}$.

Proof. $R_n = R_{1n} + R_{2n}$, $R_G = R_{1G} + R_{2G}$

$$\begin{aligned} R_{1n} &= E_*\{\exp[c_1(\phi_{1n}(x) - \theta_1)] - c_i[\phi_{1n}(x) - \theta_1] - 1\} \\ &= E_{(x^{(1)},x^{(2)},\dots,x^{(n)},x)}[\mathrm{e}^{c_1\phi_{1n}(x)}E(\mathrm{e}^{-c_1\theta_1}|x) - c_1\phi_{1n}(x) + c_1E(\theta_1|x) - 1] \\ R_{1G} &= E_{(x)}[\mathrm{e}^{c_1\phi_{1G}(x)}E(\mathrm{e}^{-c_1\theta_1}|x) - c_1\phi_{1G}(x) + c_1E(\theta_1|x) - 1] \\ &= E_{(x^{(1)},x^{(2)},\dots x^{(n)},x)}[-c_1\phi_{1G}(x) + c_1E(\theta_1|x)] \end{aligned}$$

Therefore,

$$\begin{aligned} R_{1n} - R_{1G} &= E_{(x^{(1)},x^{(2)},\dots x^{(n)},x)}\{\exp[c_1(\phi_{1n}(x) - \phi_{1G}(x))] \\ &\quad - c_1(\phi_{1n}(x) - \phi_{1G}(x)) - 1\} \\ &= \mathrm{E}_*\{\exp[c_1(\phi_{1n}(x) - \phi_{1G}(x))] - c_1(\phi_{1n}(x) - \phi_{1G}(x)) - 1\} \end{aligned}$$

Similarly,

$$\begin{aligned} R_{2n} - R_{2G} &= E_{(x^{(1)},x^{(2)},\dots x^{(n)},x)}\{\exp[c_2(\phi_{2n}(x) - \phi_{2G}(x))] \\ &\quad - [c_2(\phi_{2n}(x) - \phi_{2G}(x))] - 1\} \\ &= \mathrm{E}_*\{\exp[c_2(\phi_{2n}(x) - \phi_{2G}(x))] - [c_2(\phi_{2n}(x) - \phi_{2G}(x))] - 1\} \end{aligned}$$

Hence

$$\begin{aligned} R_n - R_G &= \sum_{i=1}^{2}(R_{\text{in}} - R_{iG}) \\ &= \sum_{i=1}^{2} E_*\{\exp[c_i(\phi_{\text{in}}(x) - \phi_{iG}(x))] - [c_i(\phi_{\text{in}}(x) - \phi_{iG}(x))] - 1\}. \quad \square \end{aligned}$$

Lemma 5. *If the conditions of Lemma* 3 *hold, then*

$$E(D_{1n}(x) - D_{1G}(x))^{2\lambda_1} \leqslant Mn^{-\frac{\lambda_1 s}{s+1}}\left(\int_{a_1}^{x_1} \frac{\Lambda_2(\theta_1, x_2) + \Lambda_1^{1/2}(\theta_1, x_2)}{u(\theta_1, x_2)}\,\mathrm{d}\theta_1\right)^{2\lambda_1} \quad \text{and}$$

$$E(D_{2n}(x) - D_{2G}(x))^{2\lambda_1} \leqslant Mn^{-\frac{\lambda_1 s}{s+1}}\left(\int_{a_2}^{x_2} \frac{\Lambda_2(x_1, \theta_2) + \Lambda_1^{1/2}(x_1, \theta_2)}{u(x_1, \theta_2)}\,\mathrm{d}\theta_2\right)^{2\lambda_1}$$

Proof

$$
\begin{aligned}
E(D_{1n}(x) - D_{1G}(x))^2 &= E\left(u(x_1, x_2)\int_{a_1}^{x_1} c_1 \mathrm{e}^{-c_1\theta_1} \frac{f_n(\theta_1, x_2) - f(\theta_1, x_2)}{u(\theta_1, x_2)}\,\mathrm{d}\theta_1\right)^2 \\
&\leqslant ME\left(\int_{a_1}^{x_1} c_1 \mathrm{e}^{-c_1\theta_1} \frac{f_n(\theta_1, x_2) - f(\theta_1, x_2)}{u(\theta_1, x_2)}\,\mathrm{d}\theta_1\right)^2 \\
&= ME\int_{a_1}^{x_1} \frac{f_n(\theta_1, x_2) - f(\theta_1, x_2)}{u(\theta_1, x_2)}\,\mathrm{d}\theta_1 \\
&\quad \cdot \int_{a_1}^{x_1} \frac{f_n(\tilde{\theta}_1, x_2) - f(\tilde{\theta}_1, x_2)}{u(\tilde{\theta}_1, x_2)}\,\mathrm{d}\tilde{\theta}_1 \\
&= M\int_{a_1}^{x_1}\int_{a_1}^{x_1} E\left[\frac{f_n(\theta_1, x_2) - f(\theta_1, x_2)}{u(\theta_1, x_2)}\right. \\
&\quad \left.\cdot \frac{f_n(\tilde{\theta}_1, x_2) - f(\tilde{\theta}_1, x_2)}{u(\tilde{\theta}_1, x_2)}\right]\mathrm{d}\theta_1\,\mathrm{d}\tilde{\theta}_1
\end{aligned}
$$

From *Hölder* inequality and C_r inequality, we get

$$
\begin{aligned}
E(D_{1n}(x) - D_{1G}(x))^2 &\leqslant M\int_{a_1}^{x_1}\int_{a_1}^{x_1} \frac{[E(f_n(\theta_1, x_2) - f(\theta_1, x_2))^2]^{1/2}}{u(\theta_1, x_2)} \\
&\quad \cdot \frac{[E(f_n(\tilde{\theta}_1, x_2) - f(\tilde{\theta}_1, x_2))^2]^{1/2}}{u(\tilde{\theta}_1, x_2)}\,\mathrm{d}\theta_1\,\mathrm{d}\tilde{\theta}_1 \\
&= M\left\{\int_{a_1}^{x_1} \frac{[E(f_n(\theta_1, x_2) - f(\theta_1, x_2))^2]^{1/2}}{u(\theta_1, x_2)}\,\mathrm{d}\theta_1\right\}^2 \\
&\leqslant M\cdot n^{-\frac{s}{s+1}}\left(\int_{a_1}^{x_1} \frac{\Lambda_2(\theta_1, x_2) + \Lambda_1^{1/2}(\theta_1, x_2)}{u(\theta_1, x_2)}\,\mathrm{d}\theta_1\right)^2
\end{aligned}
$$

Therefore

$$
E(D_{1n}(x) - D_{1G}(x))^{2\lambda_1} \leqslant Mn^{-\frac{\lambda_1 s}{s+1}}\left(\frac{\Lambda_2(\theta_1, x_2) + \Lambda_1^{1/2}(\theta_1, x_2)}{u(\theta_1, x_2)}\,\mathrm{d}\theta_1\right)^{2\lambda_1}
$$

$$
E(D_{2n}(x) - D_{2G}(x))^{2\lambda_1} \leqslant Mn^{-\frac{\lambda_1 s}{s+1}}\left(\int_{a_2}^{x_2} \frac{\Lambda_2(x_1, \theta_2) + \Lambda_1^{1/2}(x_1, \theta_2)}{u(x_1, \theta_2)}\,\mathrm{d}\theta_2\right)^{2\lambda_1}. \quad \square
$$

Lemma 6. *Let $\mu > 2$ be a given natural number. If $E_{(x)}|\tau_{iG}(x)|^{\mu} < +\infty$, for $i = 1, 2$, then $R_G < +\infty$.*

Proof. $R_G = R_{1G} + R_{2G}$

$$R_{iG} = E_{(x)}\{c_i E(\theta_i|x) - c_i\phi_{iG}(x)\} \leqslant E_{(x)}\{c_i|E(\theta_i|x)| + c_i|\phi_{iG}(x)|\}$$

$$\leqslant E_{(x)}\left\{\frac{1}{E(\mathrm{e}^{-c_i\theta_i}|x)} + \ln|\tau_{iG}(x)|\right\} \leqslant 2E_{(x)}|\tau_{iG}(x)|$$

$$\leqslant 2(E_{(x)}|\tau_{iG}(x)|^{\mu})^{1/\mu} < +\infty, \quad i = 1,2$$

Hence $R_G < +\infty$. □

Proof of Theorem. From Lemmas 6 and 4, we have $R_n - R_G = \sum_{i=1}^{2}(R_{\text{in}} - R_{iG})$

$$R_{1n} - R_{1G} = E_{(x^{(1)},\dots,x^{(n)},x)}\{\exp(c_1[\phi_{1n}(x) - \phi_{1G}(x)]) - c_1[\phi_{1n}(x) - \phi_{1G}(x)] - 1\}$$

From Lemma 1, we obtain

$$R_{1n} - R_{1G} \leqslant E_{(x^{(1)},\dots,x^{(n)},x)}[\mathrm{e}^{c_1\phi_{1n}(x)} - \mathrm{e}^{c_1\phi_{1G}(x)}]^2 = E_{(x)}\{E[\mathrm{e}^{c_1\phi_{1n}(x)} - \mathrm{e}^{c_1\phi_{1G}(x)}]^2\}$$

$$= E_{(x)}\{E(\tau_{1n}(x) - \tau_{1G}(x))^2\}$$

Let $A_{1n} = \left\{(x_1,x_2)|1 < \tau_{1G}(x) \leqslant \frac{1}{2}n^{v_1}\right\}$ and B_{1n} is complement of set A_{1n}. Therefore, when $(x_1,x_2) \in B_{1n}$, from (2.2) $(\tau_{1n}^2(x) + \tau_{1G}(x))^2 \leqslant 2(\tau_{1G}(x) + n^{2v_1}) \leqslant 10\tau_{1G}^2(x)$. From *Hölder* inequality and *Markov* inequality, we have

$$E_*\{(\tau_{1n}(x) - \tau_{1G}(x))^2 I_{B_{1n}}\} \leqslant M(E_{(x)}\left|\tau_{1G}^{\mu}(x)\right|)^{2/\mu}\left(\frac{E_{(x)}|\tau_{1G}(x)|^{\mu}}{n^{\mu v_1}}\right)^{1-(2/\mu)}$$

$$= M(E_{(x)}\left|\tau_{1G}^{\mu}(x)\right|)n^{-v_1(\mu-2)}$$

when $(x_1,x_2) \in A_{1n}$,

$$E(\tau_{1n}(x) - \tau_{1G}(x))^2 \leqslant \left(\frac{3}{2}n^{v_1}\right)^{2-2\lambda} E\left\{|\tau_{1n}(x) - \tau_{1G}(x)|_{\frac{3}{2}n^{v_1}}\right\}^{2\lambda}$$

$$= \left(\frac{3}{2}n^{v_1}\right)^{2-2\lambda} E\left\{\left|\frac{f_n(x)}{\mathrm{e}^{-c_1x_1}f_n(x) + D_{1n}(x)} - \frac{f(x)}{\mathrm{e}^{-c_1x_1}f(x) + D_{1G}(x)}\right|_{\frac{3}{2}n^{v_1}}\right\}^{2\lambda}$$

From Lemma 2 and C_r inequality, we get

$$
\begin{aligned}
E(\tau_{1n}(x)-\tau_{1G}(x))^2 &\leqslant \left(\frac{3}{2}n^{v_1}\right)^{2-2\lambda}\cdot 2(\mathrm{e}^{-c_1x_1}f(x)\\
&\quad + D_{1G}(x))^{-2\lambda}\left\{E|f_n(x)-f(x)|^{2\lambda}+\left(\frac{f(x)}{\mathrm{e}^{-c_1x_1}f(x)+D_{1G}(x)}+\frac{3}{2}n^{v_1}\right)^{2\lambda}\right.\\
&\quad \left.\times E(\mathrm{e}^{-c_1x_1}(f_n(x)-f(x))+D_{1n}(x)-D_{1G}(x))^{2\lambda}\right\}\\
&\leqslant 5n^{v_1(2-2\lambda)}(\mathrm{e}^{-c_1x_1}f(x)+D_{1G}(x))^{-2\lambda}\{E|f_n(x)-f(x)|^{2\lambda}+(2n^{v_1})^{2\lambda}\\
&\quad \times E(\mathrm{e}^{-c_1x_1}(f_n(x)-f(x)+D_{1n}(x)-D_{1G}(x))^{2\lambda}\}\\
&\leqslant 5n^{v_1(2-2\lambda)}(\mathrm{e}^{-c_1x_1}f(x)+D_{1G}(x))^{-2\lambda}\{(1+2(2n^{v_1})^{2\lambda})E(f_n(x)-f(x))^{2\lambda}\\
&\quad +2(2n^{v_1})^{2\lambda}E(D_{1n}(x)-D_{1G}(x))^{2\lambda}\}\\
&\leqslant Mn^{v_1(2-2\lambda)}(\mathrm{e}^{-c_1x_1}f(x)+D_{1G}(x))^{-2\lambda}\{n^{2\lambda u_1}E|f_n(x)-f(x)|^{2\lambda}\\
&\quad +n^{2\lambda u_1}E|D_{1n}(x)-D_{1G}(x)|^{2\lambda}\}=Mn^{2u_1}(\mathrm{e}^{-c_1x_1}f(x)\\
&\quad +D_{1G}(x))^{-2\lambda}\{E|f_n(x)-f(x)|^{2\lambda}+E|D_{1n}(x)-D_{1G}(x)|^{2\lambda}\}.
\end{aligned}
$$

From Lemmas 3 and 5, we get

$$
\begin{aligned}
E(\tau_{1n}(x)-\tau_{1G}(x))^2 &\leqslant Mn^{2v_1}[\mathrm{e}^{-c_1x_1}f(x)+D_{1G}(x)]^{-2\lambda}n^{-(\lambda s)/(s+1)}\{M_1[\Lambda_2^{2\lambda}(x)\\
&\quad +\Lambda_1^{\lambda}(x)]+M_2\left(\int_{a_1}^{x_1}\frac{\Lambda_2(\theta_1,x_2)+\Lambda_1^{1/2}(\theta_1,x_2)}{u(\theta_1,x_2)}\,\mathrm{d}\theta_1\right)^{2\lambda}\\
&\leqslant Mn^{2v_1-(\lambda s/(s+1))}[\mathrm{e}^{-c_1x_1}f(x)+D_{1G}(x)]^{-2\lambda}\\
&\quad \times\left\{\Lambda_2^{2\lambda}(x)+\Lambda_1^{\lambda}(x)+\left(\int_{a_1}^{x_1}\frac{\Lambda_2(\theta_1,x_2)+\Lambda_1^{1/2}(\theta_1,x_2)}{u(\theta_1,x_2)}\,\mathrm{d}\theta_1\right)^{2\lambda}\right\}\\
&\leqslant Mn^{2v_1-(\lambda s/(s+1))}[\mathrm{e}^{-c_1x_1}f(x)+D_{1G}(x)]^{-2\lambda}\\
&\quad \times\left\{\Lambda_2^{2\lambda}(x)+\Lambda_1^{\lambda}(x)+\left(\int_{a_1}^{x_1}\frac{\Lambda_2(\theta_1,x_2)+\Lambda_1^{1/2}(\theta_1,x_2)}{u(\theta_1,x_2)}\,\mathrm{d}\theta_1\right)^{2\lambda}\right\}
\end{aligned}
$$

Consequently,

$$E_*[(\tau_{1G}(x)-\tau_{1n}(x))^2 I_{A_{1n}}]\leqslant M\cdot n^{2v_1-(\lambda s/(s+1))}E_{(x)}\times\left\{[\mathrm{e}^{-c_1x_1}f(x)+D_{1G}(x)]^{-2\lambda}\left[\Lambda_2^{2\lambda}(x)+\Lambda_1^{\lambda}(x)+\left(\int_{a_1}^{x_1}\frac{\Lambda_2(\theta_1,x_2)+\Lambda_1^{1/2}(\theta_1,x_2)}{u(\theta_1,x_2)}\,\mathrm{d}\theta_1\right)^{2\lambda}\right]\right\}$$

When $\frac{\lambda s}{s+1}-2v_1=v_1(\mu-2)$, we get $v_1=\frac{\lambda s}{(s+1)\mu}$, $R_{1n}-R_{1G}=O(n^{-q})$, $q=\frac{\lambda s(\mu-2)}{(s+1)\mu}$. Similarly when $v_2=v_1$, we get $R_{2n}-R_{2G}=O(n^{-q})$. Hence $R_n-R_G=O(n^{-q})$, $q=\frac{\lambda s(\mu-2)}{(s+1)\mu}$. □

From the theorem, we can show that the EB estimation of parameters is asymptotically optimal. If λ is approaching 1 arbitrarily, μ and S are large enough. Then the convergence rate of the empirical Bayes estimation also approaches $O(n^{-1})$.

4. An example

We give an example to explain the reasonableness of above result. Let c_1, c_2 be 1 in loss function, $\mu>2$, and $0<\lambda<1/2$,

$$f(x_1,x_2|\theta_1,\theta_2)=\begin{cases}4\mathrm{e}^{-2(x_1-\theta_1)}\mathrm{e}^{-2(x_2-\theta_2)}, & x_i>\theta_i,\ i=1,2\\ 0, & \text{otherwise}\end{cases}$$

$$g(\theta_1,\theta_2)=\begin{cases}9\mathrm{e}^{-3\theta_1-3\theta_2}, & 0<\theta_1<+\infty,\ 0<\theta_2<+\infty\\ 0, & \text{otherwise}\end{cases}$$

It is easy to verify that the conditions of Theorem hold.

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References

[1] A.P. Basu, N. Ebrahimi, Bayesian approach to life testing and reliability estimation using asymmetric loss function, Journal of Statistical Planning and Inference 29 (1) (1991) 21–31.

[2] A. Chaturredi, M.I. Bhatti, K. Kumar, Bayesian analysis of disturbances variance in the linear regression model under asymmetric loss function, Applied Mathematics and Computation 114 (2000) 149–153.

[3] S.Y. Hunang, T.C. Liang, Empirical Bayes estimation of the truncation parameter with linex loss, Statistica Sinica 7 (3) (1997) 755–769.

[4] H. Liang, The convergence of estimation for two-dimension two-sided truncated distribution families, Mathematical Statistics and Applied Probability 4 (1991) 433–447.

[5] Shi Yimin, Empirical Bayes estimation for the parameter of two-side truncated distribution families under linex loss, Applied Mathematics. A Journal of Chinese Universities. Series A 15 (4) (2001) 475–483.

[6] H.R. Varian, A Bayesian approach to real estate assessment, in: S.E. Feinberg, A. Zellner (Eds.), Studies in Bayesian Econometrics and Statistics in Honor of L.J. Savage, North-Holland, Amsterdam, 1975, pp. 195–208.

[7] A. Zellner, Bayesian estimation and prediction using asymmetric Loss functions, Journal of the American Statistical Association 81 (2) (1986) 446–451.

[8] L.C. Zhao, Empirical Bayes estimation with convergence rates about a class of discrete distribution families, Journal of Mathematical Research and Exposition 1 (1) (1981) 59–69.