ELSEVIER

Statistics & Probability Letters 50 (2000) 23–32

STATISTICS & PROBABILITY LETTERS

www.elsevier.nl/locate/stapro

Minimax and Γ-minimax estimation for the Poisson distribution under LINEX loss when the parameter space is restricted

Alan T.K. Wan[a, *], Guohua Zou[b], Andy H. Lee[c]

[a]Department of Management Sciences, City University of Hong Kong, Tat Chee Avenue, Kowloon, Hong Kong
[b]Institute of Systems Science, Chinese Academy of Sciences, Beijing 100080, People's Republic of China
[c]Department of Epidemiology and Biostatistics, Curtin University of Technology, WA 6845, Australia

Received February 1999; received in revised form December 1999

Abstract

This paper considers the problems of minimax and Γ-minimax estimation under the LINEX loss function when the parameter space is restricted. A general property of the risk of the Bayes estimator with respect to the two-point prior is presented. Minimax and Γ-minimax estimators of the parameter of the Poisson distribution are obtained when the parameter of interest is known to lie in a small parameter space.

MSC: 62C20

Keywords: Minimax estimator; Γ-minimax estimator; LINEX loss function; Restricted parameter space; Least favorable prior; Poisson distribution

1. Introduction

Minimax estimation of unknown parameters in restricted parameter space has been a subject of interest over the past decades. Suppose that $X \sim \mathrm{N}(\theta, 1)$, $\theta \in [\alpha, \beta]$, $\beta > \alpha$; Ghosh (1964) gives a sequence of estimators of θ in the space of estimators with uniformly bounded risk whose maximum risk converges to the minimax value. Following this work, Casella and Strawderman (1981) derive the exact form of the minimax estimator of θ and the least favorable prior for the case where β–α is sufficiently small. Zinzius (1979, 1981) uses a convexity argument to investigate the problem considered by Casella and Strawderman (1981). The convexity technique is a powerful tool for finding minimax estimators. This method has been used to find minimax estimators for different distributions by many authors. See, for example, Bischoff et al. (1995a), DasGupta (1985), Eichenauer-Herrmann and Fieger (1989) and Zou (1993).

It is interesting to note that virtually all of the aforementioned studies relate only to quadratic loss. Being symmetric, the quadratic loss imposes equal penalty on over- and under-estimation of the same magnitude.

* Corresponding author.
E-mail address: msawan@cityu.edu.hk (A.T.K. Wan).

PII: S0167-7152(00)00077-8

There are situations where over- and under-estimation can lead to different consequences. For example, when estimating the average life of the components of a spaceship or an aircraft, over-estimation is usually more serious than under-estimation. In fact, Feynman's (1987) report suggests that the space shuttle disaster of 1986 was partly the result of the management's over-estimation of the average life of the solid fuel rocket booster. Zellner (1986) also suggests that in dam construction, under-estimation of the peak water level is often much more serious than over-estimation. These examples illustrate that in many situations, the quadratic loss function can be unduly restrictive and inappropriate, and suggest that we should consider properties of estimators based on an asymmetric loss function instead.

In a study of real estate assessment, Varian (1975) introduces the following asymmetric linear exponential (LINEX) loss function:

$$L(\delta,\theta) = b\{\mathrm{e}^{a(\delta-\theta)} - a(\delta-\theta) - 1\}, \tag{1.1}$$

where $a(\neq 0)$ is a shape parameter and $b > 0$ is a factor of proportionality. The LINEX loss reduces to quadratic loss for small values of a. If a is positive (negative), then over (under)-estimation is considered to be more serious than under (over)-estimation of the same magnitude, and vice versa. Numerous authors have considered the LINEX loss in various problems of interest. Examples are Zellner (1986), Parsian (1990), Takagi (1994), Cain and Janssen (1995), Ohtani (1995), Zou (1997), Wan (1999) and Wan and Kurumai (1999).

Using Zinzius's (1979, 1981) convexity technique, Bischoff et al. (1995b) obtain minimax and Γ-minimax estimators for estimating a bounded normal mean under LINEX loss. In this paper, we take their analysis further by considering the problems of minimax and Γ-minimax estimation of the parameter of the Poisson distribution under the same loss. In Section 2, we consider a general family of distributions and discuss the risk properties of the Bayes estimators of the parameters of this family of distributions. A result of Bischoff et al. (1995b) concerning the Bayes estimator is nested as a special case in our findings. In Section 3, we derive, for the Poisson distribution with sufficiently small parameter space $[0,\beta]$, the least favorable prior and minimax estimator using the convexity technique of Zinzius (1979, 1981) for the case of $n\geqslant 1$ observations. Finally, in Section 4, we consider Γ-minimax estimation of the Poisson parameter for a special type of priors.

Since b is only a factor of proportionality, we assume, without loss of generality, that $b=1$ in the subsequent analysis.

2. Preliminary results

Let $X_1,\dots,X_n$ be i.i.d. random variables, P_θ be the distribution of $X=(X_1,\dots,X_n)$ with the parameter $\theta\in[\alpha,\beta]$, $\alpha<\beta$. Assume that P_θ is dominated by some σ-finite measure μ. Further, let $f(x,\theta)$ be the Radon–Nikodym derivative of P_θ with respect to μ. We assume that $f(x,\alpha)+f(x,\beta)\neq 0$ for all $x\in\mathscr{X}$, where $\mathscr{X}$ is sample space, and $P_\theta\{x{:}f(x,\alpha)f(x,\beta)>0\}>0$ when $\theta=\alpha$ and $\theta=\beta$.

Consider the following two-point prior π:

$$\pi(\{\alpha\}) = \eta, \quad \pi(\{\beta\}) = 1-\eta, \tag{2.1}$$

where $0<\eta<1$.

It can be shown that the corresponding Bayes estimator is

$$\delta_\pi(x) = \frac{1}{a}\log\frac{\eta f(x,\alpha)+(1-\eta)f(x,\beta)}{\eta f(x,\alpha)\mathrm{e}^{-a\alpha}+(1-\eta)f(x,\beta)\mathrm{e}^{-a\beta}}. \tag{2.2}$$

Note that $\delta_\pi(x)=\beta$ if $f(x,\alpha)=0$; and $\delta_\pi(x)=\alpha$ if $f(x,\beta)=0$.

Theorem 2.1. *There exists a unique* $\eta^* \in (0,1)$ *such that*

$$R(\delta_{\pi^*},\alpha) = R(\delta_{\pi^*},\beta), \tag{2.3}$$

where π^* *is the prior distribution for* $\eta = \eta^*$. *Moreover,*

$$R(\delta_\pi,\alpha) < R(\delta_\pi,\beta) \quad \text{for } \eta \in (\eta^*,1) \tag{2.4}$$

and

$$R(\delta_\pi,\alpha) > R(\delta_\pi,\beta) \quad \text{for } \eta \in (0,\eta^*). \tag{2.5}$$

Proof. Denote

$$\mathscr{X}_1 = \{x\colon f(x,\alpha)f(x,\beta) > 0\}, \tag{2.6}$$

$$\mathscr{X}_2 = \{x\colon f(x,\alpha) = 0, f(x,\beta) \neq 0\} \tag{2.7}$$

and

$$\mathscr{X}_3 = \{x\colon f(x,\beta) = 0, f(x,\alpha) \neq 0\}. \tag{2.8}$$

It is readily seen that

$$R(\delta_\pi,\theta) = \int_{\mathscr{X}_1} (\mathrm{e}^{-a\theta}A - \log A + a\theta - 1)\,\mathrm{d}P_\theta$$

$$+[\mathrm{e}^{a(\beta-\theta)} - a(\beta-\theta) - 1]P_\theta(\mathscr{X}_2) + [\mathrm{e}^{a(\alpha-\theta)} - a(\alpha-\theta) - 1]P_\theta(\mathscr{X}_3), \tag{2.9}$$

where

$$A = \frac{\eta f(x,\alpha) + (1-\eta)f(x,\beta)}{\eta f(x,\alpha)\mathrm{e}^{-a\alpha} + (1-\eta)f(x,\beta)\mathrm{e}^{-a\beta}}. \tag{2.10}$$

Observe that $P_\alpha(\mathscr{X}_2) = P_\beta(\mathscr{X}_3) = 0$. So we have

$$R(\delta_\pi,\alpha) - R(\delta_\pi,\beta) = \int_{\mathscr{X}_1} (\mathrm{e}^{-a\alpha}A - \log A + a\alpha - 1)\,\mathrm{d}P_\alpha - \int_{\mathscr{X}_1} (\mathrm{e}^{-a\beta}A - \log A + a\beta - 1)\,\mathrm{d}P_\beta. \tag{2.11}$$

It is easily seen that if $a > 0$, then $\mathrm{e}^{a\alpha} < A < \mathrm{e}^{a\beta}$; alternatively, if $a < 0$, then $\mathrm{e}^{a\alpha} > A > \mathrm{e}^{a\beta}$. Hence, using the dominated convergence theorem, we obtain

$$\lim_{\eta\to 0^+} [R(\delta_\pi,\alpha) - R(\delta_\pi,\beta)] = [\mathrm{e}^{a(\beta-\alpha)} - a(\beta-\alpha) - 1]P_\alpha(\mathscr{X}_1) > 0 \tag{2.12}$$

and

$$\lim_{\eta\to 1^-} [R(\delta_\pi,\alpha) - R(\delta_\pi,\beta)] = -[\mathrm{e}^{a(\alpha-\beta)} - a(\alpha-\beta) - 1]P_\beta(\mathscr{X}_1) < 0. \tag{2.13}$$

Since $R(\delta_\pi,\theta)$ is continuous in η, there exists $\eta^* \in (0,1)$ such that $R(\delta_{\pi^*},\alpha) = R(\delta_{\pi^*},\beta)$.

Obviously, (2.11) can be rewritten as

$$R(\delta_\pi,\alpha) - R(\delta_\pi,\beta) = \int_{\mathscr{X}_1} \{[\mathrm{e}^{-a\alpha}f(x,\alpha) - \mathrm{e}^{-a\beta}f(x,\beta)]A - [f(x,\alpha) - f(x,\beta)]\log A\}\,\mathrm{d}\mu$$

$$+ (a\alpha - 1)P_\alpha(\mathscr{X}_1) - (a\beta - 1)P_\beta(\mathscr{X}_1). \tag{2.14}$$

Denote

$$h(\eta) = [\mathrm{e}^{-a\alpha} f(x,\alpha) - \mathrm{e}^{-a\beta} f(x,\beta)]A - [f(x,\alpha) - f(x,\beta)]\log A. \tag{2.15}$$

Then for $x \in \mathscr{X}_1$,

$$\begin{aligned} h'(\eta) &= \frac{A'}{A}\{[\mathrm{e}^{-a\alpha} f(x,\alpha) - \mathrm{e}^{-a\beta} f(x,\beta)]A - [f(x,\alpha) - f(x,\beta)]\} \\ &= \frac{A'}{A}\frac{f(x,\alpha)f(x,\beta)(\mathrm{e}^{-a\alpha} - \mathrm{e}^{-a\beta})}{\eta f(x,\alpha)\mathrm{e}^{-a\alpha} + (1-\eta)f(x,\beta)\mathrm{e}^{-a\beta}} \\ &= -\frac{1}{A}\frac{f^2(x,\alpha)f^2(x,\beta)(\mathrm{e}^{-a\alpha} - \mathrm{e}^{-a\beta})^2}{[\eta f(x,\alpha)\mathrm{e}^{-a\alpha} + (1-\eta)f(x,\beta)\mathrm{e}^{-a\beta}]^3} \\ &< 0, \end{aligned} \tag{2.16}$$

where A' is the derivative of A with respect to η. So, $h(\eta)$ is a strictly decreasing function of η, which shows from (2.14) that $R(\delta_\pi,\alpha) - R(\delta_\pi,\beta)$ is also a strictly decreasing function of η. Therefore, η^* such that $R(\delta_{\pi^*},\alpha) = R(\delta_{\pi^*},\beta)$ in $(0,1)$ is unique, and $R(\delta_\pi,\alpha) < R(\delta_\pi,\beta)$ for $\eta \in (\eta^*,1)$ and $R(\delta_\pi,\alpha) > R(\delta_\pi,\beta)$ for $\eta \in (0,\eta^*)$. This completes the proof of Theorem 2.1. □

The above theorem represents a general result which holds for a range of distributions. For example, taking $n = 1$ and $f(x,\theta) = (1/\sqrt{2\pi})\mathrm{e}^{-(1/2)(x-\theta)^2}$, i.e., $X \sim \mathrm{N}(\theta,1)$, we obtain the following results given in Bischoff et al. (1995b).

Corollary 2.1. *For the case of* $X \sim \mathrm{N}(\theta,1)$, $\theta \in [-m,m]$, *the Bayes estimator of* θ *with respect to the prior* π *is*

$$\delta_\pi(x) = \frac{1}{a}\log\frac{\eta\mathrm{e}^{-mx} + (1-\eta)\mathrm{e}^{mx}}{\eta\mathrm{e}^{-m(x-a)} + (1-\eta)\mathrm{e}^{m(x-a)}}. \tag{2.17}$$

Further, there exists a unique $\eta^* \in (0,1)$ *such that* $R(\delta_{\pi^*},-m) = R(\delta_{\pi^*},m)$, *and* $R(\delta_\pi,-m) < R(\delta_\pi,m)$ *for* $\eta \in (\eta^*,1)$ *and* $R(\delta_\pi,-m) > R(\delta_\pi,m)$ *for* $\eta \in (0,\eta^*)$.

Remark 2.1. From Bischoff et al. (1995b), we observe that for the above normal case, if m is small enough, then $\min\{R(\delta_\pi,-m), R(\delta_\pi,m)\} > \inf_{\theta\in[-m,m]} R(\delta_\pi,\theta)$. However, as we shall see later, this property does not hold in general (see Remark 3.4).

In the next section, we consider the application of Theorem 2.1 to the Poisson distribution.

3. Minimax estimation for Poisson parameter

Let $X_1,\ldots,X_n$ be i.i.d. random variables, $X_1 \sim \mathrm{Poisson}(\theta)$, $\theta \in [0,\beta]$, $\beta > 0$. Consider the following two-point prior π:

$$\pi(\{0\}) = \eta, \quad \pi(\{\beta\}) = 1-\eta, \tag{3.1}$$

where $0 < \eta < 1$.

Using (2.2), the corresponding Bayes estimator is

$$\delta_\pi(x_1,\ldots,x_n) = \begin{cases} \frac{1}{a}\log\frac{\eta+(1-\eta)\mathrm{e}^{-n\beta}}{\eta+(1-\eta)\mathrm{e}^{-(a+n)\beta}}, & x_1 = \cdots = x_n = 0, \\ \beta, & x_1 + \cdots + x_n \geqslant 1. \end{cases} \tag{3.2}$$

Lemma 3.1. *Suppose that either of the following conditions holds*:
(i) $a \geqslant -2n$, $\beta \leqslant \beta_0$, *where* $\beta_0 \in (0,+\infty)$ *is the unique root of the equation*

$$-\left(1+\frac{2a}{n}\right)e^{a\beta}+a\beta+\left(1+\frac{a}{n}\right)^2=0 \tag{3.3}$$

or
(ii) $a < -2n$ *and* $\beta \leqslant \beta_1$, *where* $\beta_1 \in (-(1/a)\log(1+a/n)^2, +\infty)$ *is the unique root of the equation*

$$-\left(1+\frac{2a}{n}\right)e^{a\beta}+a\beta+\log\left(1+\frac{a}{n}\right)^2+1=0, \tag{3.4}$$

then the risk function $R(\delta_\pi,\theta)$ *of the Bayes estimator* δ_π *is strictly convex on* $[0,\beta]$ *for every* $\eta \in (0,1)$.

Proof. It can be shown that the risk function of δ_π is

$$R(\delta_\pi,\theta)=e^{a(\beta-\theta)}-a(\beta-\theta)-1+(B-e^{a\beta})e^{-(a+n)\theta}-(\log B-a\beta)e^{-n\theta}, \tag{3.5}$$

where

$$B=\frac{\eta+(1-\eta)e^{-n\beta}}{\eta+(1-\eta)e^{-(a+n)\beta}}. \tag{3.6}$$

Hence, the second derivative of $R(\delta_\pi,\theta)$ with respect to θ is

$$R''(\delta_\pi,\theta)=a^2e^{a(\beta-\theta)}+(B-e^{a\beta})(a+n)^2e^{-(a+n)\theta}-(\log B-a\beta)n^2e^{-n\theta}. \tag{3.7}$$

Consider the cases of $a \geqslant -2n$ and $a < -2n$.
(1^0) When $a>0$, we have $1<B<e^{a\beta}$. So, from (3.7), we have

$$\begin{aligned}R''(\delta_\pi,\theta)&=e^{-(a+n)\theta}[a^2e^{a\beta+n\theta}+(B-e^{a\beta})(a+n)^2-(\log B-a\beta)n^2e^{a\theta}]\\&\geqslant e^{-(a+n)\theta}[a^2e^{a\beta}+(B-e^{a\beta})(a+n)^2-(\log B-a\beta)n^2].\end{aligned} \tag{3.8}$$

It is easy to see that the function $\phi(t)=(a+n)^2t-n^2\log t$ is strictly increasing in t when $t>n^2/(a+n)^2$. So, we have $\phi(B)>\phi(1)=(a+n)^2$. Recognizing this and using (3.8), we obtain

$$\begin{aligned}R''(\delta_\pi,\theta)&>e^{-(a+n)\theta}\{[a^2-(a+n)^2]e^{a\beta}+a\beta n^2+(a+n)^2\}\\&\hat{=}\ e^{-(a+n)\theta}\psi_1(\beta)\quad(\text{say}).\end{aligned} \tag{3.9}$$

Note that $\psi_1'(\beta)=-2a^2ne^{a\beta}-an^2(e^{a\beta}-1)<0$. Hence, $\psi_1(\beta)$ is strictly decreasing in β when $\beta>0$. On the other hand, we have

$$\lim_{\beta\to 0^+}\psi_1(\beta)=a^2 \quad\text{and}\quad \lim_{\beta\to+\infty}\psi_1(\beta)=-\infty. \tag{3.10}$$

Therefore, there exists a unique $\beta_0 \in (0,+\infty)$ such that $\psi_1(\beta_0)=0$, and $\psi_1(\beta)>\psi_1(\beta_0)=0$ for $\beta<\beta_0$. Thus, from (3.9), $R''(\delta_\pi,\theta)>0$ for $\beta \leqslant \beta_0$.

If $-2n \leqslant a < 0$, then $1>B>e^{a\beta}$, and $(a+n)^2 \leqslant n^2$. Therefore, from (3.7), we have

$$\begin{aligned}R''(\delta_\pi,\theta)&=e^{-n\theta}[a^2e^{a\beta+(n-a)\theta}+(B-e^{a\beta})(a+n)^2e^{-a\theta}-(\log B-a\beta)n^2]\\&\geqslant e^{-n\theta}[a^2e^{a\beta}+(B-e^{a\beta})(a+n)^2-(\log B-a\beta)n^2].\end{aligned} \tag{3.11}$$

It can be seen that the function $\phi(t)=(a+n)^2t-n^2\log t$ is strictly decreasing in t when $0<t<n^2/(a+n)^2$. So, we have $\phi(B)>\phi(1)=(a+n)^2$.

The remainder of the proof is similar to that of the case of $a > 0$.

(2^0) When $a < -2n$, we have $(a+n)^2 > n^2$. It is easily seen that the function $\phi(t) = (a+n)^2 t - n^2 \log t$ attains a minimum at $t = n^2/(a+n)^2$. Hence, we have

$$\phi(B) \geqslant \phi\left(\frac{n^2}{(a+n)^2}\right) = n^2 - n^2 \log \frac{n^2}{(a+n)^2} \tag{3.12}$$

and it follows from (3.11) that

$$R''(\delta_\pi, \theta) \geqslant \mathrm{e}^{-n\theta}\left\{[a^2 - (a+n)^2]\mathrm{e}^{a\beta} + a\beta n^2 + n^2 - n^2 \log \frac{n^2}{(a+n)^2}\right\}$$

$$\hat{=}\ \mathrm{e}^{-n\theta}\psi_2(\beta) \quad \text{(say)}. \tag{3.13}$$

It can be seen that the function $\psi_2(\beta)$ is strictly decreasing in β when $\beta > 0$. Moreover,

$$\psi_2\left(-\frac{1}{a}\log\left(1+\frac{a}{n}\right)^2\right) = \frac{a^2 n^2}{(a+n)^2} > 0 \tag{3.14}$$

and

$$\lim_{\beta \to +\infty} \psi_2(\beta) = -\infty. \tag{3.15}$$

So there exists a unique $\beta_1 \in (-(1/a)\log(1 + a/n)^2, +\infty)$ such that $\psi_2(\beta_1) = 0$ and $\psi_2(\beta) > \psi_2(\beta_1) = 0$ for $\beta < \beta_1$. Therefore, from (3.13), $R''(\delta_\pi, \theta) > 0$ for $\beta < \beta_1$. Note that the inequality symbol in (3.11) becomes a strict equality only when $\theta = 0$. Thus, $R(\delta_\pi, \theta)$ is strictly convex for $\beta \leqslant \beta_1$. This completes the proof of Lemma 3.1. □

Remark 3.1. In the proof of the second part of Lemma 3.1, the equality of (3.12) holds for some $\eta \in (0,1)$ when $\beta > -(1/a)\log(1 + a/n)^2$. This means that the right-hand side of (3.13) is sufficiently close to the lower bound of the right-hand side of (3.11). Note that when $\beta > -(1/a)\log(1+a/n)^2$, $\mathrm{e}^{a\beta} < n^2/(a+n)^2 < 1$ and B is a continuous function of η, so there exists a $\eta_0 \in (0,1)$ such that $B|_{\eta=\eta_0} = n^2/(a+n)^2$ (note that $\mathrm{e}^{a\beta} < B < 1$).

Remark 3.2. Note that Eq. (3.3) coincides with Eq. (3.4) for the boundary case of $a = -2n$.

Remark 3.3. It can be shown that when $a \geqslant -2n$, $R(\delta_\pi, \theta)$ is not strictly convex on $[0, \beta]$ for some $\eta \in (0,1)$ if $\beta > \beta_0$; and when $a < -2n$, $R(\delta_\pi, \theta)$ is not strictly convex on $[0, \beta]$ for some $\eta \in (0,1)$ if $\beta > \beta_1$. Thus, condition (i) or (ii) given in Lemma 3.1 is also the necessary condition for $R(\delta_\pi, \theta)$ to be strictly convex on $[0, \beta]$ for every $\eta \in (0,1)$. In fact, let $a \geqslant -2n$. If $\beta > \beta_0$, then from the proof of part (i) of Lemma 3.1, we have $\psi_1(\beta) < \psi_1(\beta_0) = 0$. Note that

$$\lim_{\theta \to 0^+, \eta \to 1^-} R''(\delta_\pi, \theta) = [a^2 - (a+n)^2]\mathrm{e}^{a\beta} + a\beta n^2 + (a+n)^2$$

$$= \psi_1(\beta) < 0. \tag{3.16}$$

So there exist $\delta > 0$ and $\varepsilon > 0$ such that

$$R''(\delta_\pi, \theta) < 0 \quad \text{for } \theta \in (0, \delta),\ \ \eta \in (1-\varepsilon, 1). \tag{3.17}$$

This illustrates that $R(\delta_\pi, \theta)$ is strictly concave in $(0, \delta)$ and hence not convex on $[0, \beta]$ for $\eta \in (1-\varepsilon, 1)$. Similarly, for the case of $a < -2n$, if $\beta > \beta_1$, then $\psi_2(\beta) < \psi_2(\beta_1) = 0$, and $\beta > -(1/a)\log(1 + a/n)^2$. From

Remark 3.1, we see that there exists a $\eta_0 \in (0,1)$ such that $B|_{\eta=\eta_0} = n^2/(a+n)^2$. Therefore,

$$\lim_{\theta\to 0^+,\eta\to\eta_0} R''(\delta_\pi,\theta) = [a^2-(a+n)^2]\mathrm{e}^{a\beta} + a\beta n^2 + n^2 - n^2\log\frac{n^2}{(a+n)^2}$$
$$= \psi_2(\beta) < 0. \tag{3.18}$$

This also shows that $R(\delta_\pi,\theta)$ is not convex on $[0,\beta]$ for some $\eta\in(0,1)$.

Lemma 3.2. *There exists a unique $\eta^*\in(0,1)$, such that*

$$R(\delta_{\pi^*},0) = R(\delta_{\pi^*},\beta), \tag{3.19}$$

where π^* is the prior distribution for $\eta=\eta^*$. Moreover, $R(\delta_\pi,0) < R(\delta_\pi,\beta)$ for $\eta\in(\eta^*,1)$ and $R(\delta_\pi,0) > R(\delta_\pi,\beta)$ for $\eta\in(0,\eta^*)$.

Proof. Lemma 3.2 is a direct result of Theorem 2.1. □

Remark 3.4. Contrary to the normal case, for the Poisson distribution, $\min\{R(\delta_\pi,0),\ R(\delta_\pi,\beta)\}$ can equal $\inf_{\theta\in[0,\beta]} R(\delta_\pi,\theta)$ for some $\eta\in(0,1)$. For example, if we let $a \geqslant -2n$, $\eta>\eta^*$, then from Lemma 3.2, we have $R(\delta_\pi,0) < R(\delta_\pi,\beta)$. Further, let $\beta\leqslant\beta_0$, then from (3.5), we have

$$R'(\delta_\pi,0) = -a\mathrm{e}^{a\beta} + a - (a+n)(B-\mathrm{e}^{a\beta}) + n(\log B - a\beta)$$
$$\hat{=}\, g_1(\eta) \quad \text{(say)}. \tag{3.20}$$

It can be shown that $g_1(\eta)$ is strictly increasing in $\eta\in(0,1)$. So if $g_1(\eta^*)\geqslant 0$, then $g_1(\eta) > g_1(\eta^*)\geqslant 0$ for $\eta>\eta^*$, which implies $R'(\delta_\pi,0)>0$ for $\eta>\eta^*$. If $g_1(\eta^*)<0$, then there exists a unique $\eta^{**}\in(\eta^*,1)$ such that $g_1(\eta^{**})=0$ (note that $\lim_{\eta\to 1^-} g_1(\eta) = n(\mathrm{e}^{a\beta}-1-a\beta)>0$). Thus, $g_1(\eta)>g_1(\eta^{**})=0$ for $\eta>\eta^{**}$, which implies $R'(\delta_\pi,0)>0$ for $\eta>\eta^{**}$. Further, note that $R''(\delta_\pi,\theta)>0$ for $\beta\leqslant\beta_0$. So we have $R'(\delta_\pi,\theta)>R'(\delta_\pi,0)>0$ when $\eta>\eta^*$ or $\eta>\eta^{**}$. Therefore, $R(\delta_\pi,0)$ is the minimum of $R(\delta_\pi,\theta)$ when $\eta>\eta^*$ or $\eta>\eta^{**}$. Similar results can be shown for $a<-2n$, $\eta>\eta^*$ and $\beta\leqslant\beta_1$.

Theorem 3.1. *Suppose that either condition* (i) *or* (ii) *of Lemma* 3.1 *holds, then the two-point prior π^*,*

$$\pi^*(\{0\}) = \eta^*, \quad \pi^*(\{\beta\}) = 1-\eta^* \tag{3.21}$$

is the least favorable prior, and the corresponding Bayes estimator,

$$\delta_{\pi^*}(x_1,\ldots,x_n) = \begin{cases} \frac{1}{a}\log\frac{\eta^*+(1-\eta^*)\mathrm{e}^{-n\beta}}{\eta^*+(1-\eta^*)\mathrm{e}^{-(a+n)\beta}}, & x_1=\cdots=x_n=0,\\ \beta, & x_1+\cdots+x_n\geqslant 1\end{cases} \tag{3.22}$$

is the minimax estimator of θ.

Proof. The proof follows from Lemmas 3.1 and 3.2. □

To illustrate our results further, we perform a simple numerical exercise to find the greatest values of β for different n for which the estimator δ_{π^*} is minimax. In Tables 1–4, β' is the upper bound of β obtained analytically; β'' is the largest value of β such that the risk function $R(\delta_{\pi^*},\theta)$ of the estimator δ_{π^*} is strictly convex; and β''' is the maximal value of β such that the estimator δ_{π^*} is minimax. Note that β' equals β_0 if $a\geqslant -2n$ and equals β_1 if $a<-2n$.

Table 1
Numerical bounds of β for various a when $n=1$

a	−5	−3	−1	−0.2	0.2	1	3	5
β'	0.789	0.906	0.567	0.468	0.433	0.378	0.293	0.244
β''	0.813	0.906	0.797	0.709	0.667	0.595	0.466	0.386
β'''	2.849	1.951	1.151	0.953	0.877	0.757	0.571	0.463

Table 2
Numerical bounds of β for various a when $n=3$

a	−5	−3	−1	−0.2	0.2	1	3	5
β'	0.234	0.189	0.161	0.152	0.148	0.141	0.126	0.115
β''	0.287	0.266	0.241	0.231	0.227	0.218	0.198	0.181
β'''	0.459	0.384	0.327	0.309	0.300	0.285	0.252	0.227

Table 3
Numerical bounds of β for various a when $n=5$

a	−5	−3	−1	−0.2	0.2	1	3	5
β'	0.113	0.102	0.094	0.091	0.089	0.087	0.081	0.076
β''	0.159	0.150	0.142	0.138	0.137	0.133	0.126	0.119
β'''	0.230	0.209	0.191	0.184	0.181	0.175	0.162	0.151

Table 4
Numerical bounds of β for various a when $n=10$

a	−5	−3	−1	−0.2	0.2	1	3	5
β'	0.05	0.048	0.046	0.045	0.045	0.044	0.042	0.041
β''	0.074	0.072	0.070	0.069	0.069	0.068	0.066	0.064
β'''	0.102	0.097	0.093	0.092	0.091	0.089	0.086	0.083

4. Γ-minimax estimation for Poisson parameter

In this section, we consider the following class of priors:

$$\Gamma_{\eta,\tau} = \{\pi = \eta\pi_1 + (1-\eta)\pi_2\colon\ \pi_1([0,\tau]) = \pi_2([\tau,\beta]) = 1\}, \tag{4.1}$$

where $\eta \in (0,1)$ and $\tau \in (0,\beta)$ are fixed. Numerous authors have used this type of priors in various contexts of interest. Examples are Lehn and Rummel (1987), Chen and Eichenauer-Herrmann (1988), Eichenauer-Herrmann et al. (1988), Bischoff and Fieger (1992) and Bischoff et al. (1995b).

It can be seen from (3.5) that

$$\begin{aligned} R'(\delta_\pi,\beta) &= -(a+n)(B-\mathrm{e}^{a\beta})\mathrm{e}^{-(a+n)\beta} + n(\log B - a\beta)\mathrm{e}^{-n\beta} \\ &\hat{=}\, g_2(B) \quad \text{(say)}. \end{aligned} \tag{4.2}$$

Obviously, when

$$a > 0 \quad \text{and} \quad 0 < \beta \leqslant \frac{1}{a}\log\left(1+\frac{a}{n}\right), \tag{4.3}$$

we have $g_2'(B) < 0$. Also, when

$$-n < a < 0 \quad \text{and} \quad 0 < \beta \leqslant \frac{1}{a}\log\left(1 + \frac{a}{n}\right) \tag{4.4}$$

or

$$a \leqslant -n, \tag{4.5}$$

we have $g_2'(B) > 0$.

Recalling that $1 < B < e^{a\beta}$ if $a > 0$ and $e^{a\beta} < B < 1$ if $a < 0$, and noting that $g_2(e^{a\beta}) = 0$, we can see that under conditions (4.3) and (4.4) or (4.5)

$$g_2(B) > g_2\left(e^{a\beta}\right) = 0, \tag{4.6}$$

which implies that $R'(\delta_\pi, \beta) > 0$. Therefore, $R(\delta_\pi, \beta) > \inf_{\theta \in [0,\beta]} R(\delta_\pi, \theta)$ if conditions (4.3) and (4.4) or (4.5) holds. Making use of this and Lemma 3.2, we see that when $0 < \eta < \eta^*$ and the conditions (4.3) and (4.4) or (4.5) holds, there exists a $\tau^* \in (0, \beta)$ such that $R(\delta_\pi, \tau^*) = R(\delta_\pi, \beta)$. In the case where $R''(\delta_\pi, \theta) > 0$ for $\theta \in (0, \beta]$, such τ^* is unique.

Further, we can show that $\beta_0 < (1/a)\log(1 + a/n)$ for $a > -n$. Thus, we obtain the following theorem.

Theorem 4.1. *Assume that either conditions* (i) *or* (ii) *of Lemma* 3.1 *holds, and* $0 < \eta < \eta^*$ *and* $\tau^* \leqslant \tau < \beta$, *then the prior* π,

$$\pi(\{0\}) = \eta, \quad \pi(\{\beta\}) = 1 - \eta \tag{4.7}$$

is least favorable in $\Gamma_{\eta,\tau}$ *and the corresponding Bayes estimator* δ_π *is* $\Gamma_{\eta,\tau}$-minimax.

Remark 4.1. Obviously, the minimax estimators and Γ-minimax estimators obtained in this paper are also admissible.

Acknowledgements

The work of the first and second authors was supported, respectively, by a strategic research grant from the City University of Hong Kong and by a grant from the National Natural Science Foundation of China. The authors also thank the referee and Dr. Wolfgang Bischoff for their useful comments and suggestions.

References

Bischoff, W., Fieger, W., 1992. Minimax estimators and Γ-minimax estimators for a bounded normal mean under the loss $l_p(\theta, d) = |\theta - d|^p$. Metrika 39, 185–197.

Bischoff, W., Fieger, W., Ochtrop, S., 1995a. Minimax estimation for the bounded mean of a bivariate normal distribution. Metrika 42, 379–394.

Bischoff, W., Fieger, W., Wulfert, S., 1995b. Minimax and Γ-minimax estimation of a bounded normal mean under LINEX loss. Statist. Decisions 13, 287–298.

Cain, M., Janssen, C., 1995. Real estate price prediction under asymmetric loss. Ann. Inst. Statist. Math. 47, 401–414.

Casella, G., Strawderman, W.E., 1981. Estimating a bounded normal mean. Ann. Statist. 9, 870–878.

Chen, L., Eichenauer-Herrmann, J., 1988. Two point priors and Γ-minimax estimation in families of uniform distributions. Statist. Papers 29, 45–57.

DasGupta, A., 1985. Bayes minimax estimation in multiparameter families when the parameter space is restricted to a bounded convex set. Sankhyā Ser. A 47, 326–332.

Eichenauer-Herrmann, J., Fieger, W., 1989. Minimax estimation in scale parameter families when the parameter interval is bounded. Statist. Decisions 7, 363–376.

Eichenauer-Herrmann, J., Kirschgarth, P., Lehn, J., 1988. Gamma-minimax estimators for a bounded normal mean. Statist. Decisions 6, 343–348.

Feynman, R.P., 1987. Mr. Feynman goes to Washington. Engineering and Science, California Institute of Technology, pp. 6–62.

Ghosh, M.N., 1964. Uniform approximation of minimax point estimates. Ann. Math. Statist. 35, 1031–1047.

Lehn, J., Rummel, F., 1987. Gamma-minimax estimation of a binomial probability under squared error loss. Statist. Decisions 5, 229–249.

Ohtani, K., 1995. Generalized ridge regression estimators under the LINEX loss function. Statist. Papers 36, 99–110.

Parsian, A., 1990. On the admissibility of an estimator of a normal mean vector under a LINEX loss function. Ann. Inst. Statist. Math. 42, 657–669.

Takagi, Y., 1994. Local asymptotic minimax risk bounds for asymmetric loss functions. Ann. Statist. 22, 39–48.

Varian, H.R., 1975. A Bayesian approach to real estate assessment. In: Fienberg, S.E., Zellner, A. (Eds.), Studies in Bayesian Econometrics and Statistics in Honour of L.J. Savage. North-Holland, Amsterdam, pp. 195–208.

Wan, A.T.K., 1999. A note on almost unbiased generalized ridge regression estimator under asymmetric loss. J. Statist. Comput. Simulation 62, 411–421.

Wan, A.T.K., Kurumai, H., 1999. An iterative feasible minimum mean squared error estimator of the disturbance variance in linear regression under asymmetric loss. Statist. Probab. Lett. 45, 253–259.

Zellner, A., 1986. Bayes estimation and prediction using asymmetric loss functions. J. Amer. Statist. Assoc. 81, 446–451.

Zinzius, E., 1979. Beiträge zur Theorie der nichtsequentiellen Parameterschätzprobleme. Doctoral Thesis, Universität Karlsruhe.

Zinzius, E., 1981. Minimaxschätzer für den Mittelwert θ einer normalverteilten Zufallsgröβe mit bekannter Varianz bei vorgegebener oberer und unterer Schranke für θ. Math. Oper. Statist. Ser. Statist. 12, 551–557.

Zou, G.H., 1993. Minimax estimation for the restricted parameter space. J. Systems Sci. Math. Sci. 13, 345–348 (in Chinese).

Zou, G.H., 1997. Admissible estimation for finite population under the LINEX loss function. J. Statist. Plann. Inference 61, 373–384.