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Admissible estimation for finite population under the Linex loss function

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Abstract

This paper considers the estimation problems of an arbitrary linear function of the characteristic values of a finite population under the Linex loss function. We obtain all admissible linear estimators when the variance σ^2 is known and all admissible linear estimators in the class of linear estimators when σ^2 is unknown.

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1. Introduction

Because the use of symmetric loss functions may be inappropriate in some practical problems, discussion of the estimation problems under asymmetric loss functions receives much attention recently (see, for example, Zellner, 1986; Bischoff et al., 1995). Varian (1975) introduced the following useful asymmetric Linex loss function:

$$L(\delta, \theta) = b(\exp\{a(\delta - \theta)\} - a(\delta - \theta) - 1), \tag{1.1}$$

where $a \neq 0$, $b > 0$ are known constants.

Zellner (1986) proved that the usual sample mean is inadmissible for estimating normal mean (in the case in which the variance is known) under the above loss function. Later, Rojo (1987) considered the admissibility of linear functions of the sample mean under the Linex loss function (1.1) and generalized Zellner's result. Bolfarine (1989) considered the estimation problems of the finite population total under the Linex loss function (at this time, θ in (1.1) means the population total). He gave the Bayes estimators of the population total and discussed the admissibility of some of the derived estimators. The objective of this paper is to investigate the admissibility of linear estimators of an arbitrary linear function of the characteristic values of a finite population under the Linex loss function.

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Suppose the finite population $\{Y_1, \ldots, Y_N\}$ is a random sample from the following superpopulation model:

$$y_k = a_k\beta + b_k + \varepsilon_k, \tag{1.2}$$

where $k = 1, \ldots, N$, $a_k > 0$ and b_k are known constants, β is unknown parameter, ε_k is normal with mean zero and variance σ^2 and $\varepsilon_1, \ldots, \varepsilon_N$ are mutually independent. This model is very useful and was discussed in detail by Cassel et al. (1976, 1977). Godambe (1982) also considered it.

We will consider the estimation problems of linear function $\sum_{k=1}^N p_k Y_k$ ($p_k > 0$, $k = 1, \ldots, N$), using the Linex loss function (1.1), under the superpopulation model (1.2). We assume that the sample $\{y_k, k \in s\}$ is drawn by an arbitrary sampling design p (i.e., $p(s)$ satisfies $p(s) > 0$, and $\sum_{s \in S} p(s) = 1$, where S is a class of subsets of $1, \ldots, N$). For the case in which σ^2 is known, we obtain all admissible linear estimators of $\sum_{k=1}^N p_k Y_k$. Because σ^2 is often unknown in the practical problems, we also investigate the admissibility of a linear estimator in this case. We obtain all admissible linear estimators of $\sum_{k=1}^N p_k Y_k$ in the class of linear estimators. Unlike under the squared error loss, for the cases in which σ^2 is known or unknown, the necessary and sufficient conditions for a linear estimator to be admissible under the Linex loss are quite different, at least in the class of linear estimators, which is somewhat surprising (see Remark 3).

The reasons why the author considers linear function $\sum_{k=1}^N p_k Y_k$ are the following:

(a) By transformation, it includes the usual case of $E(\varepsilon_k^2) = \sigma^2 a_k^g$ ($g \geqslant 0$ is a known constant).

(b) In some practical problems, it is necessary to estimate linear function $\sum_{k=1}^N p_k Y_k$ (cf., Page et al., 1993).

Since the values of b have no effect on the admissibility, we assume $b = 1$ in the Linex loss function (1.1).

2. All admissible linear estimators of $\sum_{k=1}^N p_k Y_k$ when σ^2 is known

Theorem 1. *Suppose σ^2 is known. Then the necessary and sufficient conditions for the estimator $T(s) = \sum_{k \in s} \omega_{ks} y_k + \omega_{0s}$ of linear function $\sum_{k=1}^N p_k Y_k$ to be admissible are that there exists λ_s such that $\omega_{ks} = \lambda_s a_k + p_k$ ($k \in s$), and one of the following two conditions is satisfied:*

(i) *$0 \leqslant \lambda_s < c_s/d_s$, where $c_s \triangleq \sum_{k \notin s} p_k a_k$ and $d_s \triangleq \sum_{k \in s} a_k^2$;*

(ii) *$\lambda_s = c_s/d_s$, and*

$$\omega_{0s} = -\frac{c_s}{d_s}\sum_{k \in s} a_k b_k + \sum_{k \notin s} p_k b_k - \frac{a\sigma^2}{2}\left[\left(\frac{c_s}{d_s}\right)^2 d_s + \sum_{k \notin s} p_k^2\right].$$

Proof. By linear transformation, we need only consider the case of $b_k = 0$ ($k = 1, \ldots, N$). In this case, condition (ii) becomes

$$\text{(ii)}' \qquad \lambda_s = \frac{c_s}{d_s}, \text{ and } \omega_{0s} = -\frac{a\sigma^2}{2}\left[\left(\frac{c_s}{d_s}\right)^2 d_s + \textstyle\sum_{k \notin s} p_k^2\right].$$

First it can be seen that the risk of the estimator $T(s) = \sum_{k \in s} \omega_{ks} y_k + \omega_{0s}$ is

$$R\left(T, \sum_{k=1}^{N} p_k Y_k\right) \triangleq E\left[\exp\left\{a\left(T(s) - \sum_{k=1}^{N} p_k Y_k\right)\right\} - a\left(T(s) - \sum_{k=1}^{N} p_k Y_k\right) - 1\right]$$
$$= \exp\left\{a\left[\beta\left(\sum_{k \in s} (\omega_{ks} - p_k) a_k - \sum_{k \notin s} p_k a_k\right) + \omega_{0s}\right] + \frac{a^2\sigma^2}{2}\left[\sum_{k \in s} (\omega_{ks} - p_k)^2 + \sum_{k \notin s} p_k^2\right]\right\}$$
$$- a\left[\beta\left(\sum_{k \in s} (\omega_{ks} - p_k) a_k - \sum_{k \notin s} p_k a_k\right) + \omega_{0s}\right] - 1. \tag{2.1}$$

The proof of necessity consists of the following three steps.

(1) We prove that there exists λ_s such that $\omega_{ks} = \lambda_s a_k + p_k (k \in s)$. In fact, if it is not the case, then $(\omega_{ks} - p_k)/a_k$ $(k \in s)$ are not a constant. Define

$$\omega_{ks}^* = \frac{\sum_{k \in s} (\omega_{ks} - p_k) a_k}{\sum_{k \in s} a_k^2} \cdot a_k + p_k \ (k \in s); \tag{2.2}$$
$$\omega_{0s}^* = \omega_{0s}.$$

Then we have

$$\sum_{k \in s} (\omega_{ks}^* - p_k) a_k = \sum_{k \in s} (\omega_{ks} - p_k) a_k, \tag{2.3}$$

and by the Cauchy–Schwarz inequality,

$$\sum_{k \in s} (\omega_{ks}^* - p_k)^2 - \sum_{k \in s} (\omega_{ks} - p_k)^2 = \frac{\left[\sum_{k \in s} (\omega_{ks} - p_k) a_k\right]^2}{\sum_{k \in s} a_k^2} - \sum_{k \in s} (\omega_{ks} - p_k)^2 < 0. \tag{2.4}$$

So, from (2.1), (2.3) and (2.4), the estimator $T^*(s) = \sum_{k \in s} \omega_{ks}^* y_k + \omega_{0s}^*$ is superior to $T(s)$, which contradicts the admissibility of $T(s)$.

(2) We prove $0 \leqslant \lambda_s \leqslant c_s/d_s$. Because we have shown in (1) that there exists λ_s such that $\omega_{ks} = \lambda_s a_k + p_k$ $(k \in s)$, the risk of $T(s)$ can be expressed as

$$R\left(T, \sum_{k=1}^{N} p_k Y_k\right) = \exp\left\{a[\beta(\lambda_s d_s - c_s) + \omega_{0s}] + \frac{a^2\sigma^2}{2}\left(\lambda_s^2 d_s + \sum_{k \notin s} p_k^2\right)\right\}$$
$$- a[\beta(\lambda_s d_s - c_s) + \omega_{0s}] - 1 \triangleq f(\lambda_s, \omega_{0s}). \tag{2.5}$$

Let ω_{0s} be a function of λ_s such that its derivative with respect to λ_s satisfies

$$\frac{d\omega_{0s}}{d\lambda_s} = \left[\omega_{0s} + \frac{a\sigma^2}{2}\left(\lambda_s^2 d_s + \sum_{k \notin s} p_k^2\right)\right] \Big/ \left(\lambda_s - \frac{c_s}{d_s}\right). \tag{2.6}$$

(Eq. (2.6) always has solutions since it is a linear first-order differential equation.) Then from (2.5) and (2.6), we get

$$\frac{\mathrm{d}f}{\mathrm{d}\lambda_s} = a\left(\beta d_s + \frac{\mathrm{d}\omega_{0s}}{\mathrm{d}\lambda_s}\right)\left[\exp\left\{a\left(\beta d_s\left(\lambda_s - \frac{c_s}{d_s}\right) + \omega_{0s} + \frac{a\sigma^2}{2}\left(\lambda_s^2 d_s + \sum_{k\notin s} p_k^2\right)\right)\right\} - 1\right]$$
$$+ a^2\sigma^2\lambda_s d_s \exp\left\{a\left(\beta d_s\left(\lambda_s - \frac{c_s}{d_s}\right) + \omega_{0s} + \frac{a\sigma^2}{2}\left(\lambda_s^2 d_s + \sum_{k\notin s} p_k^2\right)\right)\right\}$$
$$= a\left(\beta d_s + \frac{\mathrm{d}\omega_{0s}}{\mathrm{d}\lambda_s}\right)\left[\exp\left\{a\left(\beta d_s + \frac{\mathrm{d}\omega_{0s}}{\mathrm{d}\lambda_s}\right)\left(\lambda_s - \frac{c_s}{d_s}\right)\right\} - 1\right] + a^2\sigma^2\lambda_s d_s \cdot \exp\left\{a\left(\beta d_s + \frac{\mathrm{d}\omega_{0s}}{\mathrm{d}\lambda_s}\right)\left(\lambda_s - \frac{c_s}{d_s}\right)\right\}. \tag{2.7}$$

Note that if $\lambda_s < 0$, then

$$a\left(\beta d_s + \frac{\mathrm{d}\omega_{0s}}{\mathrm{d}\lambda_s}\right)\left[\exp\left\{a\left(\beta d_s + \frac{\mathrm{d}\omega_{0s}}{\mathrm{d}\lambda_s}\right)\left(\lambda_s - \frac{c_s}{d_s}\right)\right\} - 1\right] \leqslant 0. \tag{2.8}$$

So, $\mathrm{d}f/\mathrm{d}\lambda_s < 0$ when $\lambda_s < 0$. That is, $f(\lambda_s)$ is a strictly decreasing function of λ_s when $\lambda_s < 0$. Thus, increasing λ_s will reduce the risk of $T(s)$. This shows that $T(s)$ with $\lambda_s < 0$ is not admissible.

Similarly, $f(\lambda_s)$ is a strictly increasing function of λ_s when $\lambda_s > c_s/d_s$. Therefore, $T(s)$ with $\lambda_s > c_s/d_s$ is not admissible either.

(3) Now we prove that if $\lambda_s = c_s/d_s$, then

$$\omega_{0s} = -\frac{a\sigma^2}{2}\left[\left(\frac{c_s}{d_s}\right)^2 d_s + \sum_{k\notin s} p_k^2\right].$$

This conclusion can be obtained readily: it follows from (2.5) that when $\lambda_s = c_s/d_s$,

$$R\left(T, \sum_{k=1}^{N} p_k Y_k\right) = \exp\left\{a\omega_{0s} + \frac{a^2\sigma^2}{2}\left[\left(\frac{c_s}{d_s}\right)^2 d_s + \sum_{k\notin s} p_k^2\right]\right\} - a\omega_{0s} - 1, \tag{2.9}$$

which attains its minimum only at

$$\omega_{0s} = -\frac{a\sigma^2}{2}\left[\left(\frac{c_s}{d_s}\right)^2 d_s + \sum_{k\notin s} p_k^2\right].$$

Summarizing (1)–(3), necessity is proved.

In order to prove sufficiency, we consider the following three cases.

(1) Assume $\lambda_s = 0$. At this time, the risk of $T(s)$ is

$$R\left(T, \sum_{k=1}^{N} p_k Y_k\right) = \exp\left\{a(-\beta c_s + \omega_{0s}) + \frac{a^2\sigma^2}{2}\sum_{k\notin s} p_k^2\right\} - a(-\beta c_s + \omega_{0s}) - 1. \tag{2.10}$$

Now we suppose that the estimator $\delta(y_k, k \in s)$ satisfies

$$R\left(\delta, \sum_{k=1}^{N} p_k Y_k\right) \leqslant R\left(T, \sum_{k=1}^{N} p_k Y_k\right) \quad \text{for all } \beta. \tag{2.11}$$

Note that

$$\begin{aligned} R\left(\delta, \sum_{k=1}^{N} p_k Y_k\right) &= E\left[\exp\left\{a\left(\delta - \sum_{k=1}^{N} p_k Y_k\right)\right\} - a\left(\delta - \sum_{k=1}^{N} p_k Y_k\right) - 1\right] \\ &= E\exp\left\{a\left[\delta - \left(\sum_{k\in s} p_k y_k + \omega_{0s}\right)\right]\right\} \\ &\quad \times E\exp\left\{-a\left(\sum_{k\notin s} p_k y_k - \omega_{0s}\right)\right\} \\ &\quad - aE\left[\delta - \left(\sum_{k\in s} p_k y_k + \omega_{0s}\right)\right] + aE\left(\sum_{k\notin s} p_k Y_k - \omega_{0s}\right) - 1 \\ &= E\exp\left\{a\left[\delta - \left(\sum_{k\in s} p_k y_k + \omega_{0s}\right)\right]\right\}\exp\left\{a(-\beta c_s + \omega_{0s})\right. \\ &\quad \left. + \frac{a^2\sigma^2}{2}\sum_{k\notin s} p_k^2\right\} - aE\left[\delta - \left(\sum_{k\in s} p_k y_k + \omega_{0s}\right)\right] \\ &\quad + a(\beta c_s - \omega_{0s}) - 1. \end{aligned} \tag{2.12}$$

So, from (2.10) and (2.12), we can see that (2.11) is equivalent to

$$\begin{aligned} &E\exp\left\{a\left[\delta - \left(\sum_{k\in s} p_k y_k + \omega_{0s}\right)\right]\right\}\exp\left\{a(-\beta c_s + \omega_{0s}) + \frac{a^2\sigma^2}{2}\sum_{k\notin s} p_k^2\right\} \\ &\quad - aE\left[\delta - \left(\sum_{k\in s} p_k y_k + \omega_{0s}\right)\right] \leqslant \exp\left\{a(-\beta c_s + \omega_{0s}) + \frac{a^2\sigma^2}{2}\sum_{k\notin s} p_k^2\right\}. \end{aligned} \tag{2.13}$$

Taking

$$\beta = \left(\omega_{0s} + \frac{a\sigma^2}{2}\sum_{k\notin s} p_k^2\right)\Big/ c_s \triangleq \beta_0$$

in (2.13), we get

$$E_{\beta_0}\left(\exp\left\{a\left[\delta-\left(\sum_{k\in s}p_ky_k+\omega_{0s}\right)\right]\right\}-a\left[\delta-\left(\sum_{k\in s}p_ky_k+\omega_{0s}\right)\right]-1\right)\leqslant 0, \tag{2.14}$$

where E_{β_0} denotes expectation when the parameter is β_0. Since the integrand in (2.14) is nonnegative, we have $\delta=\sum_{k\in s}p_ky_k+\omega_{0s}=T(s)$ (a.e. Lebesgue). Therefore, $T(s)$ is admissible.

(2) Assume $0<\lambda_s<c_s/d_s$. Let β have the prior distribution $\mathrm{N}(\mu,\tau^2)$ (where μ, $\tau^2>0$ are known). After some calculations, we can obtain the corresponding Bayes estimator to be

$$\delta_{\mathrm{B}}(y_k,k\in s)=\sum_{k\in s}\left(\frac{\tau^2c_s}{\sigma^2+\tau^2d_s}a_k+p_k\right)y_k+\frac{\mu\sigma^2c_s-a\sigma^2\tau^2c_s^2/2}{\sigma^2+\tau^2d_s}-\frac{a\sigma^2}{2}\sum_{k\notin s}p_k^2. \tag{2.15}$$

Therefore, when $0<\lambda_s<c_s/d_s$, the estimator $T(s)=\sum_{k\in s}(\lambda_sa_k+p_k)y_k+\omega_{0s}$ is the Bayes estimator with respect to some prior distribution $\mathrm{N}(\mu_0,\tau_0^2)$. Since the loss function (1.1) is strictly convex, $T(s)$ is the unique Bayes estimator and hence admissible.

(3) When $\lambda_s=c_s/d_s$, and

$$\omega_{0s}=-\frac{a\sigma^2}{2}\left[\left(\frac{c_s}{d_s}\right)^2d_s+\sum_{k\notin s}p_k^2\right],$$

by using the limiting Bayes method (see Lehmann, 1983, p. 265 or Rojo, 1987), we can show that $T(s)$ is admissible. In fact, from (2.5), we have

$$R\left(T,\sum_{k=1}^Np_kY_k\right)=\frac{a^2\sigma^2}{2}\left[\left(\frac{c_s}{d_s}\right)^2d_s+\sum_{k\notin s}p_k^2\right]\triangleq r. \tag{2.16}$$

Suppose that the estimator $\delta^*(y_k,k\in s)$ is superior to $T(s)$, then

$$R\left(\delta^*,\sum_{k=1}^Np_kY_k\right)\leqslant R\left(T,\sum_{k=1}^Np_kY_k\right)\quad\text{for all }\beta, \tag{2.17}$$

$$R\left(\delta^*,\sum_{k=1}^Np_kY_k\right)< R\left(T,\sum_{k=1}^Np_kY_k\right)\quad\text{for some }\beta_0. \tag{2.18}$$

Using the fact that $R(\delta^*,\sum_{k=1}^Np_kY_k)$ is a continuous function of β, we can find an $\varepsilon>0$ and $\beta_1<\beta_2$ such that

$$R\left(\delta^*,\sum_{k=1}^Np_kY_k\right)< R\left(T,\sum_{k=1}^Np_kY_k\right)-\varepsilon,\qquad\text{for all }\beta_1<\beta<\beta_2. \tag{2.19}$$

Let now δ'_{B} be the Bayes estimator with respect to the prior distribution $\mathrm{N}(0, \tau^2)$, and let $B(\delta'_{\mathrm{B}})$ be the Bayes risk of δ'_{B}. Then from (2.15), we have

$$\delta'_{\mathrm{B}}(y_k, k \in s) = \sum_{k \in s}\left(\frac{\tau^2 c_s}{\sigma^2 + \tau^2 d_s} \cdot a_k + p_k\right) y_k - \frac{a\sigma^2}{2}\left(\frac{\tau^2 c_s^2}{\sigma^2 + \tau^2 d_s} + \sum_{k \notin s} p_k^2\right), \quad (2.20)$$

and from (2.5), its Bayes risk is

$$B(\delta'_{\mathrm{B}}) = \frac{a^2\sigma^2}{2}\left(\frac{\tau^2 c_s^2}{\sigma^2 + \tau^2 d_s} + \sum_{k \notin s} p_k^2\right). \quad (2.21)$$

Let $B(\delta^*)$ be the Bayes risk of the estimator δ^* with respect to the prior distribution $\mathrm{N}(0, \tau^2)$. Then from (2.16), (2.17), (2.19) and (2.21), we get

$$\frac{r - B(\delta^*)}{r - B(\delta'_{\mathrm{B}})} = \frac{\int_{-\infty}^{\infty} [R(T, \sum_{k=1}^N p_k Y_k) - R(\delta^*, \sum_{k=1}^N p_k Y_k)] \frac{1}{\sqrt{2\pi}\tau} \mathrm{e}^{-\beta^2/2\tau^2} \mathrm{d}\beta}{\frac{a^2\sigma^2}{2}\left[\left(\frac{c_s}{d_s}\right)^2 d_s - \frac{\tau^2 c_s^2}{\sigma^2 + \tau^2 d_s}\right]}$$

$$\geqslant \frac{2 d_s(\sigma^2 + \tau^2 d_s)\varepsilon}{\sqrt{2\pi}\tau a^2 \sigma^4 c_s^2} \int_{\beta_1}^{\beta_2} \mathrm{e}^{-\beta^2/2\tau^2} \mathrm{d}\beta \to +\infty, \quad (2.22)$$

when $\tau \to +\infty$. Thus, if τ is sufficiently large, then $B(\delta^*) < B(\delta'_{\mathrm{B}})$, which contradicts the fact that δ'_{B} is the Bayes estimator with respect to the prior distribution $\mathrm{N}(0, \tau^2)$. Therefore, $T(s)$ is admissible.

From (1)–(3), sufficiency is proved. This completes the proof of Theorem 1. □

Remark 1. For the regression superpopulation model through the origin considered by Bolfarine (1989)

$$y_k = x_k \beta + \varepsilon_k, \quad (2.23)$$

where ε_k is normal with mean zero and variance $\sigma^2 x_k$ and $\varepsilon_1, \ldots, \varepsilon_N$ are mutually independent. By making transformation $z_k = y_k/\sqrt{x_k}$ and taking $\lambda_s = c_s/d_s$,

$$\omega_{0s} = -\frac{a\sigma^2}{2}\left[\left(\frac{c_s}{d_s}\right)^2 d_s + \sum_{k \notin s} p_k^2\right],$$

we can see that $T(s) = \sum_{k \in s}(\lambda_s \sqrt{x_k} + p_k) z_k + \omega_{0s}$ is an admissible estimator of $\sum_{k=1}^N p_k Z_k$. Further, we take $p_k = \sqrt{x_k}$, then the estimator

$$T(s) = \frac{\sum_{k=1}^N X_k}{\sum_{k \in s} x_k} \sum_{k \in s} y_k - \frac{a\sigma^2}{2} \frac{\sum_{k=1}^N X_k \sum_{k \notin s} x_k}{\sum_{k \in s} x_k} \quad (2.24)$$

is an admissible estimator of the population total $\sum_{k=1}^N Y_k$. The estimator $T(s)$ in (2.24) is just the estimator $\hat{T}_{RL}$ in Bolfarine (1989), whose admissibility was also shown by Bolfarine (1989).

3. All admissible linear estimators of $\sum_{k=1}^N p_k Y_k$ in the class of linear estimators when σ^2 is unknown

Denote the class of linear estimators by $\mathscr{T}$.

Theorem 2. *Suppose σ^2 is unknown. Then the necessary and sufficient conditions for the estimator $T(s) = \sum_{k\in s} \omega_{ks} y_k + \omega_{0s}$ of linear function $\sum_{k=1}^N p_k Y_k$ to be admissible in the class $\mathscr{T}$ are that there exists λ_s such that $\omega_{ks} = \lambda_s a_k + p_k (k \in s)$, and one of the following two conditions is satisfied*:

(i) $$\left|\lambda_s - \frac{c_s}{d_s}\right| \leqslant \sqrt{\left(\frac{c_s}{d_s}\right)^2 + \frac{\sum_{k\notin s} p_k^2}{d_s}}, \quad \lambda_s \neq \frac{c_s}{d_s};$$

(ii) $$\lambda_s = \frac{c_s}{d_s}, \text{ and } a\omega_{0s} \leqslant -a\left(\frac{c_s}{d_s}\sum_{k\in s} a_k b_k - \sum_{k\notin s} p_k b_k\right).$$

Proof. As in the proof of Theorem 1, we assume $b_k = 0$ $(k = 1, \ldots, N)$. In this case, condition (ii) becomes

(ii)′ $\lambda_s = c_s/d_s$, and $a\omega_{0s} \leqslant 0$.

Necessity: From the proof of Theorem 1, we can see that if the estimator $T(s)$ is admissible in the class $\mathscr{T}$, then there must exist λ_s such that $\omega_{ks} = \lambda_s a_k + p_k$ $(k \in s)$, even though σ^2 is unknown. In the following we will prove that λ_s and ω_{0s} satisfy (i) or (ii)′.

First it is easy to see that if $\lambda_s = c_s/d_s$, then $a\omega_{0s} \leqslant 0$. Otherwise, assume $a\omega_{0s} > 0$. Then the estimator

$$T^0(s) = \sum_{k\in s}\left(\frac{c_s}{d_s}\cdot a_k + p_k\right) y_k$$

is superior to $T(s)$, a contradiction to the admissibility of $T(s)$.

Now we show that if $\lambda_s \neq c_s/d_s$, then

$$\left|\lambda_s - \frac{c_s}{d_s}\right| \leqslant \sqrt{\left(\frac{c_s}{d_s}\right)^2 + \frac{\sum_{k\notin s} p_k^2}{d_s}}. \tag{3.1}$$

If it is not the case, then the opposite inequality holds strictly. Define

$$\lambda_s' = \frac{\lambda_s c_s + \sum_{k\notin s} p_k^2}{\lambda_s d_s - c_s};$$

$$\omega_{0s}' = \frac{\lambda_s' d_s - c_s}{\lambda_s d_s - c_s}\cdot \omega_{0s}. \tag{3.2}$$

It can be shown that the corresponding estimator $T'(s) = \sum_{k\in s}(\lambda'_s a_k + p_k)y_k + \omega'_{0s}$ is superior to $T(s)$, which contradicts the admissibility of $T(s)$. In fact, let

$$t \triangleq \frac{\lambda'_s d_s - c_s}{\lambda_s d_s - c_s}, \qquad A \triangleq a[\beta(\lambda_s d_s - c_s) + \omega_{0s}], \qquad B^2 \triangleq \frac{a^2\sigma^2}{2}\left(\lambda_s^2 d_s + \sum_{k\notin s} p_k^2\right).$$

Then from (3.2), $0 < t < 1$, and from (2.5), we have

$$\begin{aligned} \Delta &\triangleq R\left(T', \sum_{k=1}^{N} p_k Y_k\right) - R\left(T, \sum_{k=1}^{N} p_k Y_k\right) \\ &= [e^{t(A+B^2)} - tA] - (e^{A+B^2} - A). \end{aligned} \tag{3.3}$$

It is easy to see that Δ is a strictly increasing function of t when $t > 0$. Hence $\Delta < 0$.

Sufficiency. From the proof of the necessity of Theorem 1, we can see that in order to show the admissibility of the estimator $T(s) = \sum_{k\in s}(\lambda_s a_k + p_k)y_k + \omega_{0s}$, it is enough to prove that there are no estimators of the form $T^*(s) = \sum_{k\in s}(\lambda_s^* a_k + p_k)y_k + \omega_{0s}^*$ superior to it.

From (2.5), $R(T^*, \sum_{k=1}^{N} p_k Y_k) \leqslant R(T, \sum_{k=1}^{N} p_k Y_k)$ if and only if

$$\begin{aligned} &\exp\left\{a[\beta(\lambda_s^* d_s - c_s) + \omega_{0s}^*] + \frac{a^2\sigma^2}{2}\left(\lambda_s^{*2} d_s + \sum_{k\notin s} p_k^2\right)\right\} - a[\beta(\lambda_s^* d_s - c_s) + \omega_{0s}^*] \\ &\leqslant \exp\left\{a[\beta(\lambda_s d_s - c_s) + \omega_{0s}] + \frac{a^2\sigma^2}{2}\left(\lambda_s^2 d_s + \sum_{k\notin s} p_k^2\right)\right\} \\ &\quad - a[\beta(\lambda_s d_s - c_s) + \omega_{0s}]. \end{aligned} \tag{3.4}$$

(1) Assume that condition (i) holds. From Theorem 1, for the case in which σ^2 is known, the estimator $T(s) = \sum_{k\in s}(\lambda_s a_k + p_k)y_k + \omega_{0s}$ is admissible when $0 \leqslant \lambda_s < c_s/d_s$. Clearly, for the case in which σ^2 is unknown, $T(s)$ is also admissible at this time. So, it suffices to consider the cases of $c_s/d_s \leqslant \lambda_s \leqslant c_s/d_s + \sqrt{(c_s/d_s)^2 + \sum_{k\notin s} p_k^2/d_s}$ and $c_s/d_s - \sqrt{(c_s/d_s)^2 + \sum_{k\notin s} p_k^2/d_s} \leqslant \lambda_s < 0$.

Let $t^* \triangleq (\lambda_s^* d_s - c_s)/(\lambda_s d_s - c_s)$. By taking $\beta = -\omega_{0s}/(\lambda_s d_s - c_s)$ in (3.4), we obtain

$$\begin{aligned} &\exp\left\{a(-t^*\omega_{0s} + \omega_{0s}^*) + \frac{a^2\sigma^2}{2}\left(\lambda_s^{*2} d_s + \sum_{k\notin s} p_k^2\right)\right\} - a(-t^*\omega_{0s} + \omega_{0s}^*) \\ &\leqslant \exp\left\{\frac{a^2\sigma^2}{2}\left(\lambda_s^2 d_s + \sum_{k\notin s} p_k^2\right)\right\}. \end{aligned} \tag{3.5}$$

Letting $\sigma^2 \to 0$ in (3.5), we get

$$\omega_{0s}^* = t^*\omega_{0s}. \tag{3.6}$$

Substituting (3.6) in (3.5), we have $\lambda_s^{*2} \leqslant \lambda_s^2$. From this and the hypothesis condition on λ_s, we can see that if $\lambda_s^* \neq \lambda_s$, then

$$\lambda_s^{*2} d_s + \sum_{k\notin s} p_k^2 > t^*\left(\lambda_s^2 d_s + \sum_{k\notin s} p_k^2\right). \tag{3.7}$$

Now taking

$$\beta = -\left[\omega_{0s} + \frac{a\sigma^2}{2}\left(\lambda_s^2 d_s + \sum_{k\notin s} p_k^2\right)\right] \Big/ (\lambda_s d_s - c_s)$$

in (3.4) and using (3.6), we have

$$\exp\left\{\frac{a^2\sigma^2}{2}\left[\left(\lambda_s^{*2} d_s + \sum_{k\notin s} p_k^2\right) - t^*\left(\lambda_s^2 d_s + \sum_{k\notin s} p_k^2\right)\right]\right\}$$
$$\leqslant 1 + (1 - t^*)\frac{a^2\sigma^2}{2}\left(\lambda_s^2 d_s + \sum_{k\notin s} p_k^2\right). \tag{3.8}$$

However, from (3.7), if $\lambda_s^* \neq \lambda_s$, then the right-hand side of (3.8) goes to $+\infty$ much slower than the left-hand side of (3.8) when $\sigma^2 \to +\infty$, which is impossible. So we must have $\lambda_s^* = \lambda_s$. From this and (3.6), $\omega_{0s}^* = \omega_{0s}$. Thus, $T(s)$ is admissible.

(2) Now assume that condition (ii)′ holds. At this time, $\lambda_s = c_s/d_s$, so (3.4) is equivalent to

$$\exp\left\{a[\beta(\lambda_s^* d_s - c_s) + \omega_{0s}^*] + \frac{a^2\sigma^2}{2}\left(\lambda_s^{*2} d_s + \sum_{k\notin s} p_k^2\right)\right\}$$
$$\leqslant \exp\left\{a\omega_{0s} + \frac{a^2\sigma^2}{2}\left[\left(\frac{c_s}{d_s}\right)^2 d_s + \sum_{k\notin s} p_k^2\right]\right\} - a\omega_{0s} + a[\beta(\lambda_s^* d_s - c_s) + \omega_{0s}^*]. \tag{3.9}$$

If $\lambda_s^* \neq c_s/d_s$, then the right-hand side of (3.9) goes to $+\infty$ much slower than the left-hand side of (3.9) when $\beta a(\lambda_s^* d_s - c_s) \to +\infty$, which is impossible. Hence $\lambda_s^* = c_s/d_s$. Substituting it in (3.9), we obtain

$$\exp\left\{a\omega_{0s}^* + \frac{a^2\sigma^2}{2}\left[\left(\frac{c_s}{d_s}\right)^2 d_s + \sum_{k\notin s} p_k^2\right]\right\} - a\omega_{0s}^*$$
$$\leqslant \exp\left\{a\omega_{0s} + \frac{a^2\sigma^2}{2}\left[\left(\frac{c_s}{d_s}\right)^2 d_s + \sum_{k\notin s} p_k^2\right]\right\} - a\omega_{0s}. \tag{3.10}$$

If $a\omega_{0s} = 0$, then by letting $\sigma^2 \to 0$ on the two sides of (3.10), we get $a\omega_{0s}^* = 0$. If $a\omega_{0s} < 0$, then by taking

$$\frac{a^2\sigma^2}{2}\left[\left(\frac{c_s}{d_s}\right)^2 d_s + \sum_{k\notin s} p_k^2\right] = -a\omega_{0s}$$

on the two sides of (3.10), we have

$$e^{a\omega_{0s}^* - a\omega_{0s}} - (a\omega_{0s}^* - a\omega_{0s}) - 1 \leqslant 0, \tag{3.11}$$

which implies $a\omega_{0s}^* = a\omega_{0s}$, that is, $\omega_{0s}^* = \omega_{0s}$. Thus, $T^*(s) = T(s)$, which shows that $T(s)$ is admissible. This completes the proof of Theorem 2. □

Remark 2. From the proof of Theorem 1, we can see that when σ^2 is known, a linear estimator is admissible in the class of all estimators if and only if it is admissible in the class of linear estimators. When σ^2 is unknown, whether the similar conclusion holds (i.e., whether the conditions in Theorem 2 are also the necessary and sufficient conditions for a linear estimator to be admissible in the class of all estimators) is an interesting problem.

Remark 3. It can be verified that the necessary and sufficient conditions for the estimator $T(s) = \sum_{k \in s} \omega_{ks} y_k + \omega_{0s}$ of linear function $\sum_{k=1}^{N} p_k Y_k$ to be admissible in the class of linear (or all) estimators under the squared error loss function (of course, for the case in which the class of linear estimators is considered, the assumption on the distribution of ε_k is unnecessary) are that there exists λ_s such that $\omega_{ks} = \lambda_s a_k + p_k (k \in s)$, and one of the following two conditions is satisfied:

(i) $0 \leqslant \lambda_s < c_s/d_s$,

(ii) $\lambda_s = \dfrac{c_s}{d_s}$ and $\omega_{0s} = -\dfrac{c_s}{d_s} \sum_{k \in s} a_k b_k + \sum_{k \notin s} p_k b_k$,

whenever σ^2 is known or unknown. Comparing this conclusion with Theorems 1 and 2, we can see that when σ^2 is known, the necessary and sufficient conditions for a linear estimator to be admissible under the Linex loss are very similar to those under the squared error loss. Actually, they are almost exactly the same unless $\lambda_s = c_s/d_s$. But for the case in which σ^2 is unknown, the necessary and sufficient conditions for a linear estimator to be admissible under the two losses are quite different, at least in the class of linear estimators, which is somewhat surprising.

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