On the Error Function of a Complex Argument

By JOSEPH KESTIN, Providence, R. I., USA[1]), and
LEIF N. PERSEN, Trondheim, Norway[2])

Introduction

The error function of a complex argument and allied functions occur in many problems of physics or engineering. They occur, for example, in the theory of heat conduction, particularly when non-steady heat conduction through stratified media is concerned as well as in problems of electro-chemical diffusion. Among the additional fields of application, electrical transients, electromagnetic theory, and rocket ballistics may be mentioned. The application of the error function to problems in statistics and of the allied Fresnel integrals to classical diffraction problems is also very well known.

In spite of their great usefulness, the tabulation of the error and related functions of a complex argument has not progressed very far because of their complexity as well as because of the large amount of numerical work involved. ROSSER [1][3]) made a comprehensive study of the methods of integrating the functions

$$\int_0^z \exp(-\xi^2)\, d\xi \quad \text{and} \quad \exp(z^2)\int_z^\infty \exp(-\xi^2)\, d\xi$$

and prepared suitable auxiliary tables. CLEMMOW and Miss MUNFORD [2] gave a short table of values of the function

$$G(z) = \exp\left(\frac{1}{2}\, i\, \pi\, z^2\right)\int_{z\sqrt{\pi/2}}^\infty \exp(-i\,\xi^2)\, d\xi$$
$$= \sqrt{\frac{1}{2}\,\pi}\, \exp\left(\frac{1}{2}\, i\, \pi\, z^2\right)\int_z^\infty \exp\left(-\frac{1}{2}\, i\, \pi\, \zeta^2\right) d\zeta\,.$$

SKWIRZYNSKI [3] prepared a short table of the complementary error function

$$\operatorname{erfc} z = \frac{2}{\sqrt{\pi}}\int_z^\infty \exp(-\xi^2)\, d\xi = \frac{2\,i}{\sqrt{\pi}}\int_{i\infty}^{i z} \exp \zeta^2\, d\zeta\,.$$

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[1]) Brown University.

[2]) Norges tekniske högskole.

[3]) Numbers in brackets refer to References, page 39.

In all these papers the integration has been performed by more or less direct methods, principally by a series expansion. Recently CAHILL of the National Bureau of Standards [4] has undertaken a programme of computation with the aid of SEAC by the general code for evaluating confluent hypergeometric functions, making use of the identity

$$\operatorname{erf} z = \frac{2\,z}{\sqrt{\pi}}\,{}_1F_1\left(\frac{1}{2},\ \frac{3}{2},\ -z^2\right).$$

Finally it should be mentioned that LAIBLE [5] prepared a relief chart of the error function

$$\operatorname{erf} z = \frac{2}{\sqrt{\pi}}\int_0^z \exp(-\xi^2)\,d\xi\,,$$

calculated the first five zeros, gave an asymptotic expression for the zeros, and indicated a graphical method for the tracing of lines of constant modulus and phase angle.

The preceding methods, apart from being tedious and time-consuming, do not lead to simple expressions for the real and imaginary parts of the function.

In the present note we wish to give a transformation which seems to us to lead to a simple expression for the related function

$$K(z) = z\exp(z^2)\operatorname{erfc}(z)\,. \tag{1}$$

Explicit formulae for the real and imaginary parts of this function

$$K(z) = K_r(x,\,y) + i\,K_i(x,\,y) \tag{2}$$

$$z = x + i\,y \tag{2a}$$

are obtained in terms of elementary functions and two definite integrals. The resulting formulae facilitate integration along lines passing through the origin, i.e. for z varying at a constant phase angle.

The Transformation

Recently the authors encountered a problem involving the calculation of slow torsional oscillations of a disk [6] and of bodies of revolution [7] where the transformation now to be explained rendered good service.

Let α and $\overline{\alpha}$ denote two complex conjugate numbers and consider the two identities:

$$\left.\begin{aligned}\frac{1}{\sqrt{s}+\alpha} - \frac{1}{\sqrt{s}+\overline{\alpha}} &\equiv \frac{\overline{\alpha}-\alpha}{(\sqrt{s}+\alpha)\,(\sqrt{s}+\overline{\alpha})}\,,\\ \frac{1}{\sqrt{s}+\alpha} + \frac{1}{\sqrt{s}+\overline{\alpha}} &\equiv \frac{2\sqrt{s}+(\alpha+\overline{\alpha})}{(\sqrt{s}+\alpha)\,(\sqrt{s}+\overline{\alpha})}\,.\end{aligned}\right\} \tag{3}$$

Putting

$$\alpha = a + i\,b\,, \quad \bar{\alpha} = a - i\,b\,, \tag{4}$$

we can rewrite the identities (3) to yield

$$\left.\begin{aligned} D(s) &= \frac{1}{\sqrt{s}+\alpha} - \frac{1}{\sqrt{s}+\bar{\alpha}} \equiv -\frac{2\,i\,b}{(\sqrt{s}+\alpha)\,(\sqrt{s}+\bar{\alpha})}\,, \\ S(s) &= \frac{1}{\sqrt{s}+\alpha} + \frac{1}{\sqrt{s}+\bar{\alpha}} \equiv \frac{2\sqrt{s}+2\,a}{(\sqrt{s}+\alpha)\,(\sqrt{s}+\bar{\alpha})}\,. \end{aligned}\right\} \tag{5}$$

We now take the inverse Laplace transforms of the left- and right-hand sides of equations (5). The left-hand sides, denoted by the subscript l can be integrated by reference to tables [8], [9], and we have

$$\mathfrak{L}^{-1}\left\{\frac{1}{\sqrt{s}+\alpha}\right\} = \frac{1}{\sqrt{\pi\,t}} - \alpha\,F\big(\alpha\sqrt{t}\big)\,, \tag{6}$$

where

$$F(z) = \exp(z^2)\,\mathrm{erfc}\,(z)\,. \tag{7}$$

Employing the notation

$$\mathfrak{L}^{-1}\{D(s)\} = D(t)\,,\ \text{etc.}\,,$$

we have

$$D_l(t) = \bar{\alpha}\,F\big(\bar{\alpha}\sqrt{t}\big) - \alpha\,F\big(\alpha\sqrt{t}\big)\,, \quad S_l(t) = \frac{2}{\sqrt{\pi\,t}} - \alpha\,F\big(\alpha\sqrt{t}\big) + \bar{\alpha}\,F\big(\bar{\alpha}\sqrt{t}\big)\,. \tag{8}$$

The right-hand sides of the identities in equations (5) can now be integrated directly noting that the integrands have simple poles at $s_1 = \alpha^2$ and $s_2 = \bar{\alpha}^2$ and branch points at the origin. The residues will be expressed by exponentials, and hence will lead to damped harmonic oscillations. The contributions from the branch points will be given in the form of definite integrals, as shown in the Appendix.

Performing the somewhat lengthy but standard computations, we find

$$\left.\begin{aligned} K_r\big(\alpha\sqrt{t}\big) = \frac{1}{\sqrt{\pi}} &+ 2\sqrt{t}\exp[(a^2-b^2)\,t]\{a\cos(2\,a\,b\,t) - b\sin(2\,a\,b\,t)\} \\ &+ (b^2-a^2)\sqrt{t}\,P_1(a,b,t) - \sqrt{t}\,P_2(a\,b\,t) \end{aligned}\right\} \tag{9}$$

and

$$\left.\begin{aligned} K_i\big(\alpha\sqrt{t}\big) = &+ 2\sqrt{t}\exp[(a^2-b^2)\,t]\{a\sin(2\,a\,b\,t) + b\cos(2\,a\,b\,t)\} \\ &+ 2\,a\,b\sqrt{t}\,P_1(a,b,t)\,, \end{aligned}\right\} \tag{9a}$$

where

$$P_1(a, b, t) = \frac{1}{\pi} \int_0^\infty \frac{\xi^{1/2} \exp(-\xi t)\, d\xi}{\xi^2 + 2\,(a^2 - b^2)\,\xi + (a^2 + b^2)^2}\,, \tag{10}$$

$$P_2(a, b, t) = \frac{1}{\pi} \int_0^\infty \frac{\xi^{3/2} \exp(-\xi t)\, d\xi}{\xi^2 + 2\,(a^2 - b^2)\,\xi + (a^2 + b^2)^2} \tag{10a}$$

all the quantities now being real. It should be noted that the integrands in equations (10) and (10a) are always positive. The calculation leads, of course, also to an expression for $K(\bar{\alpha}\sqrt{t})$ which, however, need not be considered as it follows from $K(\alpha\sqrt{t})$ by substituting $\bar{\alpha}$ for α.

Alternative Forms

In order to obtain formulae for $K(z)$, with $z = x + i\,y$, we put

$$b = 1\,, \quad \sqrt{t} = y\,, \quad a = \frac{x}{y}\,, \tag{11}$$

so that $x + i\,y = \alpha\sqrt{t}$. Hence

$$\left.\begin{aligned} K_r(x, y) = {} & \frac{1}{\sqrt{\pi}} + 2 \exp(x^2 - y^2)\left\{ x \cos(2\,x\,y) - y \sin(2\,x\,y) \right\} \\ & + \frac{y^2 - x^2}{y} P_1(x, y) - y\,P_2(x, y)\,, \end{aligned}\right\} \tag{12}$$

$$\left.\begin{aligned} K_i(x, y) = {} & + 2 \exp(x^2 - y^2)\left\{ x \sin(2\,x\,y) + y \cos(2\,x\,y) \right\} \\ & + 2\,x\,P_1(x, y)\,, \end{aligned}\right\} \tag{12a}$$

$$P_1(x, y) = \frac{1}{\pi} \int_0^\infty \frac{\xi^{1/2} \exp(-\xi\,y^2)\, d\xi}{\xi^2 + 2\,[(x/y)^2 - 1]\,\xi + [(x/y)^2 + 1]^2}\,, \tag{12b}$$

$$P_2(x, y) = \frac{1}{\pi} \int_0^\infty \frac{\xi^{3/2} \exp(-\xi\,y^2)\, d\xi}{\xi^2 + 2\,[(x/y)^2 - 1]\,\xi + [(x/y)^2 + 1]^2}\,. \tag{12c}$$

It is, of course, sufficient to perform calculations in the first quadrant, as the representations in the remaining quadrants are mirror images with respect to the axes of co-ordinates. The zeros of $K(z)$, which are identical with those of erfc z are given by solving the two simultaneous equations

$$K_r(x, y) = 0\,, \quad K_i(x, y) = 0\,. \tag{13}$$

For large values of t, i.e. for large values of $r = \sqrt{x^2 + y^2}$ the terms with the functions P_1 and P_2 may be neglected. Hence the asymptotic zeros are found from the system

$$x \cos(2\,x\,y) - y \sin(2\,x\,y) = -\frac{\exp(y^2 - x^2)}{2\sqrt{\pi}}\,, \qquad \tan(2\,x\,y) = -\frac{y}{x}\,. \quad (13\text{a})$$

The forms (12) to (12c) give a better insight into the behaviour of the function $K(z)$ and hence into that of $F(z)$ and $\operatorname{erfc}(z)$ than the standard formulae because the oscillation appears explicitly and not under the integral sign. Corresponding expressions for the real and imaginary parts of $F(z)$, $\operatorname{erfc}(z)$, and $\operatorname{erf}(z)$ can be easily deduced.

Expressions in polar co-ordinates are obtained by putting

$$z = r\,e^{i\varphi}, \quad \text{i.e.} \quad a = \cos\varphi; \quad b = \sin\varphi; \quad \sqrt{t} = r\,. \tag{14}$$

Hence

$$\left.\begin{aligned} K_r(r,\varphi) = \frac{1}{\sqrt{\pi}} &+ 2\,r\exp(r^2\cos 2\,\varphi)\cos(\varphi+\psi) \\ &- r\,P_2(r,\varphi) - r(\cos 2\,\varphi)\,P_1(r,\varphi)\,, \end{aligned}\right\} \tag{14a}$$

$$K_i(r,\varphi) = +\,2\,r\exp(r^2\cos 2\,\varphi)\sin(\varphi+\psi) + r(\sin 2\,\varphi)\,P_1(r,\varphi)\,, \tag{14b}$$

where

$$\psi = r^2 \sin 2\,\varphi\,, \tag{14c}$$

$$P_1(r,\varphi) = \frac{1}{\pi}\int_0^\infty \frac{\xi^{1/2}\exp(-r^2\,\xi)\,d\xi}{\xi^2 + 2\,\xi\cos 2\,\varphi + 1}\,, \tag{14d}$$

$$P_2(r,\varphi) = \frac{1}{\pi}\int_0^\infty \frac{\xi^{3/2}\exp(-r^2\,\xi)\,d\xi}{\xi^2 + 2\,\xi\cos 2\,\varphi + 1}\,. \tag{14e}$$

The Fresnel integral is related to the value of $K(z)$ for $\varphi = \pi/4$. In this particular case

$$K_r\left(r, \frac{\pi}{4}\right) = \frac{1}{\sqrt{\pi}} + 2\,r\cos\left(\frac{\pi}{4} + r^2\right) - r\,\frac{1}{\pi}\int_0^\infty \frac{\xi^{3/2}\exp(-r^2\,\xi)\,d\xi}{\xi^2+1}\,,$$

$$K_i\left(r, \frac{\pi}{4}\right) = +\,2\,r\sin\left(\frac{\pi}{4} + r^2\right) + r\,\frac{1}{\pi}\int_0^\infty \frac{\xi^{1/2}\exp(-r^2\,\xi)\,d\xi}{\xi^2+1}\,.$$

APPENDIX

Derivation of the Result Used in Equations (9) and (9a)

We now proceed to sketch the method of finding the inverse Laplace transforms of the right-hand sides of equations (3), i.e. to calculate D_r and S_r.

From the first of equations (5), we have

$$D_r(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} - \frac{2 i b e^{st} ds}{(\sqrt{s}+\alpha)(\sqrt{s}+\bar{\alpha})} . \tag{A-1}$$

The integrand has simple poles at $s_1 = \alpha^2$ and $s_2 = \bar{\alpha}^2$ and a branch point at the origin. The residues are:

at $s_1 = \alpha^2$

$$R(s_1) = -2\alpha \exp(\alpha^2 t) , \tag{A-2}$$

at $s_2 = \bar{\alpha}^2$

$$R(s_2) = +2\bar{\alpha} \exp(\bar{\alpha}^2 t) . \tag{A-3}$$

Taking a cut along the real negative axis in the s-plane, and putting $s = \xi e^{i\Theta}$, we find that

$$\left(\sqrt{s}+\alpha\right)\left(\sqrt{s}+\bar{\alpha}\right) = \xi \exp(i\,\Theta) + (\alpha+\bar{\alpha})\,\xi^{1/2} \exp\left(\frac{1}{2}\,i\,\Theta\right) + \alpha\bar{\alpha}$$

for $\Theta = -\pi$:

$$\left(\sqrt{s}+\alpha\right)\left(\sqrt{s}+\bar{\alpha}\right) = -\xi - (\alpha+\bar{\alpha})\,\xi^{1/2}\, i + \alpha\bar{\alpha} ,$$

for $\Theta = +\pi$:

$$\left(\sqrt{s}+\alpha\right)\left(\sqrt{s}+\bar{\alpha}\right) = -\xi + (\alpha+\bar{\alpha})\,\xi^{1/2}\, i + \alpha\bar{\alpha} ,$$

and the contribution from the branch point becomes

$$B_1 = -\frac{1}{2\pi i} \int_{\infty}^{0} - \frac{2 i b \exp(-\xi t)\, d\xi}{(-\xi+\alpha\bar{\alpha}) - i(\alpha+\bar{\alpha})\,\xi^{1/2}}$$

$$- \frac{1}{2\pi i} \int_{0}^{\infty} - \frac{2 i b \exp(-\xi t)\, d\xi}{(-\xi+\alpha\bar{\alpha}) + i(\alpha+\bar{\alpha})\,\xi^{1/2}} .$$

Evaluating and substituting $\alpha = a + i b$ and $\bar{\alpha} = a - i b$, we obtain

$$B_1 = -\frac{4 a b}{\pi}\, i \int_0^{\infty} \frac{\xi^{1/2} \exp(-\xi t)\, d\xi}{\xi^2 + 2(a^2-b^2)\,\xi + (a^2+b^2)^2} . \tag{A-4}$$

Thus

$$D_r(t) = 2\bar{\alpha}\exp(\bar{\alpha}^2 t) - 2\alpha\exp(\alpha^2 t) - \frac{4 a b i}{\pi} \int_0^{\infty} \frac{\xi^{1/2}\exp(-\xi t)\, d\xi}{\xi^2 + 2(a^2-b^2)\,\xi + (a^2+b^2)^2} . \tag{A-5}$$

Substituting for α and $\bar{\alpha}$ we can transform this to

$$D_r(t) = -4\,i\left\{\exp[(a^2-b^2)\,t][a\sin(2\,a\,b\,t)+b\cos(2\,a\,b\,t)]+a\,b\,P_1\right\}, \quad \text{(A-6)}$$

where P_1 was given in equation (10).

Proceeding in a similar manner, we find

$$S_r(t) = \frac{1}{2\,\pi\,i}\int_{c-i\infty}^{c+i\infty}\frac{(2\sqrt{s}+2\,a)\exp(s\,t)\,ds}{(\sqrt{s}+\alpha)(\sqrt{s}+\bar{\alpha})}. \quad \text{(A-7)}$$

Making use of the preceding results, it can be shown that

$$\left.\begin{aligned} S_r(t) = &-4\exp[(a^2-b^2)\,t][a\cos(2\,a\,b\,t)-b\sin(2\,a\,b\,t)] \\ &-2\,(b^2-a^2)\,P_1+2\,P_2, \end{aligned}\right\} \quad \text{(A-8)}$$

where $P_2(a, b, t)$ was given in equation (10a). Equating $D_l = D_r$ and $S_l = S_r$ we are led to the result in equations (9) and (9a).

REFERENCES

[1] J. B. Rosser, *Theory and Application of* $\int_0^z e^{-x^2}\,dx$ *and* $\int_0^z e^{-p^2y^2}\,dy\int_0^y e^{-x^2}\,dx$, Rep. OSRD 5861 prepared by Allegany Ballistics Laboratory, Contr. OEMsr-273, 1945.

[2] P. C. Clemmow and C. M. Munford, *A Table of* $\sqrt{\left(\frac{1}{2}\pi\right)}\,e^{\frac{1}{2}i\pi x^2}\int_\varrho^\infty e^{-\frac{1}{2}i\pi x^2}\,dx$ *for Complex Values of x*, Phil. Trans. [A] *245*, 189 (1952).

[3] W. Skwirzynski, *Evaluation of* erfc(z), Marconi's Wireless Telegraph Co., Res. Division, Rep. RD. 992, Great Baddow, Essex, 1952 (Unpublished).

[4] W. F. Cahill, *A Short Table of the Error Function of Complex Arguments*, NBS Project 1102–10–1104, Report 3034, Washington, 1953.

[5] T. Laible, *Höhenkarte des Fehlerintegrals*, ZAMP *2*, 484 (1951).

[6] J. Kestin and L. N. Persen, *Slow Oscillations of an Infinite Plate and an Infinite Disk in a Viscous Fluid*, Brown University Report AF-891/1, Providence, R. I., 1954 [Air Res. & Dev. Command Contract AF18(600)-891].

[7] J. Kestin and L. N. Persen, *Small Oscillations of Bodies of Revolution in a Viscous Fluid*, Brown University Report AF-891/2, Providence, R. I., 1954 [Air Res. & Dev. Command Contract AF18(600)-891].

[8] G. Doetsch, *Tabellen zur Laplace-Transformation und Anleitung zum Gebrauch* (Berlin 1947).

[9] H. Bateman, Manuscript project, *Tables of Integral Transforms* (McGraw-Hill, New York 1954).

Zusammenfassung

In der Einleitung werden die wichtigsten bisherigen Arbeiten über die Fehlerfunktion mit komplexem Argument kurz besprochen. Die der vorliegenden Untersuchung unterworfene Funktion $K(z)$ ist in Gleichung (1) definiert. Mit Hilfe der Laplace-Transformation und ihrer Umkehrformel wird gezeigt, wie sich die zwei Identitäten in Gleichung (3) behandeln lassen, so dass sowohl der Real- als auch der Imaginärteil der Funktion $K(z)$ sich in zwei Teile aufspalten lässt, was aus den Gleichungen (9) und (9a) hervorgeht. Der erste Teil kann durch elementare Funktionen ausgedrückt werden, während der zweite Teil sich durch zwei Integrale darstellen lässt. Für die praktische Anwendung der angegebenen Ausdrücke sind in den Gleichungen (12), (12a) und (14a), (14b) die Ausdrücke in kartesischen bzw. polaren Koordinaten umgeschrieben worden. Der Vorteil der angegebenen Aufspaltung liegt darin, dass die in den Ausdrücken auftretenden Integrale monoton abnehmende Funktionen der unabhängigen Veränderlichen (x, y) darstellen und sich deswegen leicht numerisch ausrechnen lassen. Der Schwingungsanteil der Funktion $K(z)$ ist ausschliesslich durch elementare Funktionen ausgedrückt. Im Appendix ist der Rechnungsvorgang, der zu den angegebenen Ausdrücken führt, näher umschrieben.

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Äquivalenzsatz, Ähnlichkeitssätze für schallnahe Geschwindigkeiten und Widerstand nicht angestellter Körper kleiner Spannweite

Von FRIEDRICH KEUNE und KLAUS OSWATITSCH, Stockholm[1])

Die erste Arbeit über die Übertragbarkeit von theoretischen oder experimentellen Resultaten an nicht angestellten Rotationskörpern auf Körper kleiner Spannweite stammt von WARD [1][2]). Er bewies mit Hilfe der Laplace-Transformationen und des Impulssatzes, dass der Widerstand eines Körpers kleiner Spannweite in Überschallströmung innerhalb des Linearisierungsbereiches der Differentialgleichungen gleich ist dem Widerstand eines Rotationskörpers gleicher Querschnittverteilung $Q(x)$. Solche Körper gleicher Verteilung der Querschnittflächen-Inhalte (Fig. 1) sind der Angelpunkt aller folgenden Überlegungen. Sie seien als «äquivalente Körper» kleiner Spannweite[3]) bezeichnet. Eine wesentliche Voraussetzung des Satzes von WARD liegt in der Forderung, dass die Querschnittänderung am Körperende $x = x_0$ verschwinden muss $[Q_x(x_0) = 0]$, das heisst der äquivalente Rotationskörper muss entweder zylindrisch oder kegelig enden (Figur $2a$ und $2c$).

Unabhängig von WARD und auf anderem Wege wurden Körper kleiner Spannweite im Linearisierungsbereich der Unter- und Überschallströmung von

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[1]) Kungl. Tekniska Högskolan, Institution Gasdynamik.
[2]) Die Ziffern in eckigen Klammern [ ] verweisen auf das Literaturverzeichnis auf Seite 63.
[3]) Vergleiche auch Figur 3.