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CONCEPTS OF TURBULENCE AND C.F.D. APPLICATIONS

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Abstract—The article's main theme is the 'turbulence modelling' used in present days' computational schemes. This modelling is an effort to account for the influence of turbulent motion on diffusion processes, and reflects our efforts to bridge what is generally recognized as the 'closure gap', i.e. to replace a needed phenomenological relation(s) with hypotheses of some kind. In this way one tries to remedy our lack of knowledge of turbulent motion. For that purpose a number of ideas and concepts over the past 60 years is mentioned and their incorporation in numerical analysis is discussed. The main emphasis is placed on the physical concepts and their consequences.

CONTENTS

1. THE STARTING POINT

Modern studies of turbulent fluid flow must be said to have begun with the deduction of the Navier–Stokes–Reynolds equations using the concept of an average flow field.[1] In doing so Reynolds introduced the concept of a physical quantity having a mean value and a turbulent fluctuation, the instantaneous value of the quantity being the sum of the two. However, his approach also incorporates the assumption that an equation retains its validity after having been subjected to the averaging procedure. More hidden may be the assumption that the Navier–Stokes equations and our other conservation laws contain all necessary information for a complete determination of the fluid flow in all its nonstationary three-dimensionality. This assumption has gained substantial support from results obtained through numerical solutions of these basic equations and their comparison with experimental results. The author met with this 'evidence' in the striking presentation by Fromm[2] at the IUTAM conference in Ann Arbor 1964, and Fig. 1 shows an example of such a comparison. It should be mentioned in passing that the conclusion is not at all obvious, note the following passage by Sommerfeld.[3]

*"Unter *B* treten wir an die Grundfrage heran: Reichen diese Gleichungen (Navier–Stokes) aus um die Beobahtungstatsachen zu erklären?" "Wir haben schon S.109 unserer Meinung Ausdruck gegeben, dass diese Grundfrage zu bejahen sei. Unsere Meinung wird aber nicht allgemein geteilt."

*"Under *B* we approach the fundamental question: Are these equations (Navier–Stokes) sufficient to explain the experimental evidence? We have already p.109 expressed our opinion, that the question is to be answered in the affirmative. Our opinion is however not generally shared."

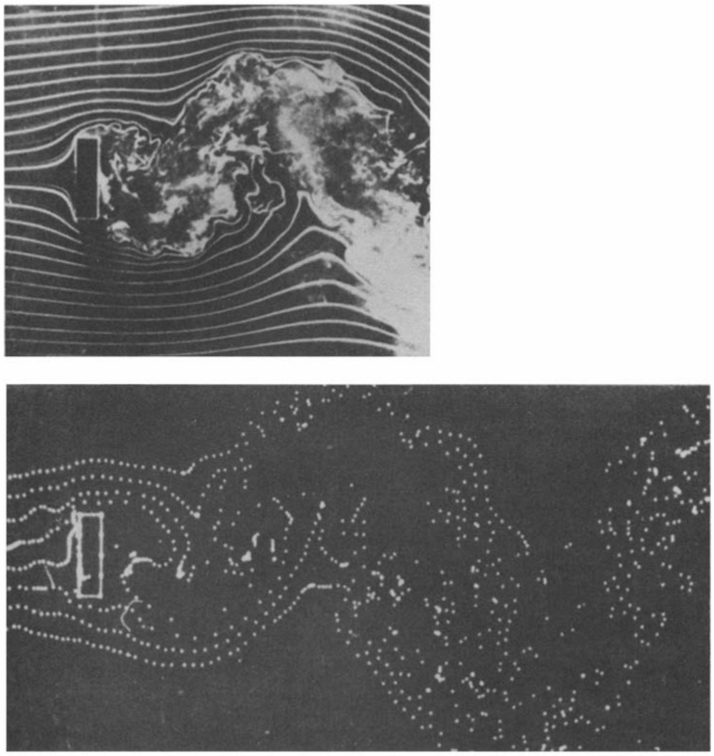

FIG. 1. Smoke channel visualisation of the flow around an obstacle and the numerical counterpart.

Having however accepted these basic concepts and assumptions one is left with the following set of equations for the mean motion (velocity components $\bar{u}, \bar{v}, \bar{w}$) of an incompressible fluid in a turbulent field.

Equation of continuity:

$$\frac{\partial \bar{u}}{\partial x}+\frac{\partial \bar{v}}{\partial y}+\frac{\partial \bar{w}}{\partial z}=0. \tag{1}$$

Equation of motion:

$$\left.\begin{aligned}
\rho\frac{D\bar{u}}{Dt}&=\rho X-\frac{\partial p}{\partial x}+\frac{\partial}{\partial x}[\bar{\sigma}_x-\rho\overline{u'u'}]+\frac{\partial}{\partial y}[\bar{\tau}_{xy}-\rho\overline{u'v'}]+\frac{\partial}{\partial z}[\bar{\tau}_{xz}-\rho\overline{u'w'}]\\
\rho\frac{D\bar{v}}{Dt}&=\rho Y-\frac{\partial p}{\partial y}+\frac{\partial}{\partial x}[\bar{\tau}_{xy}-\rho\overline{u'v'}]+\frac{\partial}{\partial y}[\bar{\sigma}_y-\rho\overline{v'v'}]+\frac{\partial}{\partial z}[\bar{\tau}_{yz}-\rho\overline{v'w'}]\\
\rho\frac{D\bar{w}}{Dt}&=\rho Z-\frac{\partial p}{\partial z}+\frac{\partial}{\partial x}[\bar{\tau}_{xz}-\rho\overline{u'w'}]+\frac{\partial}{\partial y}[\bar{\tau}_{yz}-\rho\overline{v'w'}]+\frac{\partial}{\partial z}[\bar{\sigma}_z-\rho\overline{w'w'}]
\end{aligned}\right\} \tag{2}$$

($\bar{\sigma}_x, \ldots, \bar{\tau}_{xy}, \ldots$ represent the mean viscous stress components acting on the surface of a fluid element, p is the pressure, $\rho\overline{u'u'}, \ldots, \rho\overline{u'v'}, \ldots$ represent the Reynolds apparent stresses and X, Y, Z, are body forces).

These equations are really more formidable than the ones from which they are derived because of the appearance of the Reynolds apparent stresses, and the closure problem associated with them: However this is where we start, even today.

2. THE MIXING LENGTH

Some 20 years after Reynolds deduced the Eqs (2), Prandtl[4] introduced his concept of a boundary layer, and this type of shear flow was to be the playground for testing out new ideas and concepts aimed at solving the closure problem. The concept of a mixing length was born. Because of its importance a closer exposition of the concept will be given. Prandtl starts out by defining the so-called 'turbulent ball' as a body of fluid which for a short period keeps its identity as it moves and then disintegrates and mixes with other parts of the fluid to form a new 'ball'. He creates in this way a picture of what may be conceived of as 'the structure of turbulence' and left a legacy that once a proper understanding of the structure of turbulence was established, the main problems were solved.

Prandtl's concept led him to his expression for the turbulent shear stress which in two dimensional shear flow reads

$$\tau = \rho l^2 \frac{\partial \bar{u}}{\partial y}\left|\frac{\partial \bar{u}}{\partial y}\right| \tag{3}$$

where l is the mixing length. Prandtl went however two steps further: He assumed τ to be constant through the boundary layer ($=\tau_w$) and l to be proportional to the wall distance ($=\kappa y$). This leads immediately to the well known relationship

$$u^+ = \frac{1}{\kappa} \ln y^+ + C \tag{4}$$

where u^+ and y^+ are defined as usual. $[u^+ = \bar{u}/v_*,\ y^+ = y\, v_*/\nu,\ v_*^2 = \tau_w/\rho]$.

At this point it may be of interest to notice that Eq. (4) was deduced on the assumption that it would be valid near the wall whereas the application of it has sometimes been extended to the outer region of the turbulent boundary layer.

Prandtl's approach is often referred to as a 'momentum transport theory'; alternatively Taylor[5] proposed his 'vorticity transport theory' to arrive at the mixing length concept. It is worth noting that von Karman[6] at about the same time advanced his concept of 'local similarity' to arrive at his formulation of the mixing length.

From these developments the following conclusions can be drawn: Prandtl's, Taylor's and von Karman's approach all give reasonable results for the velocity profile in shear flow over a certain range in spite of the fact that they are based on entirely different physical concepts. The lesson to be learned is therefore that a successful determination of the flow field seems not to allow the conclusion that the underlying physical assumptions are necessarily correct.

The mixing length concept seems to indicate an introduction of a 'turbulent transport coefficient'. Thus, if Eq. (3) is rewritten as

$$\tau = \rho \nu_e \frac{\partial \bar{u}}{\partial y} \tag{5}$$

where ν_e is the 'eddy viscosity',[7] a concept has been introduced which eventually leads to a general statement:

> "A flux in a turbulent field may be expressed as a gradient to a 'Driving field' multiplied by a 'turbulent transport coefficient'."

This is a generalization which one may exemplify by a turbulent 'Fourier' law stating:

$$\dot{\mathbf{q}} = -k_t \cdot \text{grad}\ \bar{T} \tag{6}$$

where $\dot{\mathbf{q}}$ is the turbulent heat flux, the driving force is the gradient to the mean temperature field and k_t is the 'eddy conductivity'. This is a very central concept in modern turbulence research, since 'turbulence modelling' makes extended use of it. As such it deserves closer scrutiny.

First, Eq. (5) is a simplification obtained by a combination of the boundary layer concept and the Boussinesq hypothesis. However, in general terms Eq. (5) represents an attempt to correlate a state of stress to a state of strain rate (velocity field) which means correlating two tensors. It seems unreasonable to give the eddy viscosity concept a wider range of validity by generalizing it to three-dimensional cases without experimental substantiation. However, this point will be considered further in the subsequent discussion of turbulence models.

Secondly, Eq. (6) is in its nature an attempt to correlate a flux and a gradient, i.e. correlation of two vectors. To assume that this in general can be obtained through the same 'turbulent transport coefficient' which sometimes is taken proportional to the eddy viscosity seems to be a rather far-fetched assumption. Again, further discussion of this point is deferred until later.

3. THE STATISTICAL APPROACH

The difficulties with the closure problem led to a search for a new concept which might lead out of the apparent dead end. The turbulent fluctuations, once they were 'readily available' due to hot wire anemometry, invited the idea of conceiving of turbulence as a stochastic process. This opened up a series of new concepts like probability density functions and frequency spectrum for the different stochastic variables, and of several correlation functions to mention just a few. These concepts, well known in statistical theories, gave new insight into turbulent fluid flow because they permit questions to be asked (and sometimes answered) which go beyond what can be expected by the purely analytic approach. One may for a moment contemplate the situation which would arise if the closure problem could be solved and a complete solution to the fundamental Eqs (1) and (2) were available: Would then all questions have been answered; questions like the probability that a given physical quantity would deviate from its mean value by more than a certain percentage? The answer is obviously negative and the conclusion must be drawn that a proper description of a turbulent flow field must go beyond what the analytic approach has to offer. Such a proper description would have to contain information on the statistical nature of all stochastic variables. The analytic 'solution' would have to be supplemented with information about intermittency, probability density functions, skewness and flatness factors, frequency spectra etc. for different physical quantities and their fluctuations.

The statistical approach has however, in addition to the concepts already mentioned, introduced two concepts of rather profound importance as far as our understanding of turbulent flows is concerned. The first concept is that of a turbulent length scale and the second is expressed through the generally accepted idea that turbulent energy is cascaded down from large eddies to smaller ones where it is finally converted to heat through viscous action. To the author's knowledge any attempt to utilize these ideas in numerical schemes is only to be found in a few investigations like Magnussen's[8] attempt to describe numerically flows with chemical reactions (flames).

4. FLOW VISUALIZATION

Flow visualization has been thought of as an excellent tool to explore the secrets of turbulent fluid flow. This idea was first used by Reynolds[9] in his famous tube experiment, and since then a series of concepts have emerged from such studies. The concept of a horseshoe vortex was introduced by Theodorsen, and Weske[10] elaborated on this concept in a series of studies. The latest concept derived from flow visualization are the 'sweeps' and 'bursts' in the boundary layer studied extensively by Brodkey *et al.*[11]

With the rapid development within the field of data acquisition and handling and the computer assisted experimentation the possibility to 'visualize electronically' has emerged and 'coherent structures' have become a concept within the study of turbulent flows. Because of the great hopes attached to this concept, a closer look at this development seems justified. Expositions by Hussain(12) and Landahl(13) are here drawn to the attention of the reader. In this connection it ought to be observed that Kline *et al.*(14) have indicated a mechanism of propagation of identifiable structures and Persen(15) has elaborated on this idea.

Flow visualization has also been used to study 'the structure of turbulence' in the vicinity of fixed walls where wall coatings were used. Kline *et al.*(14) described vortices in the viscous sublayer, and followed up the investigations of Persen(16) based on the earlier observations of Ginoux.(17) This structure has a remarkable influence on the heat transfer from a stagnation point, but has very little influence on the average shear stress (see Ref. (18)).

These studies and many more on analogous topics seem not to have been incorporated in numerical codes so far.

5. COHERENT STRUCTURES

As already mentioned the concept of 'coherent structures' has attracted attention and induced great hopes of leading to a break-through in turbulence research.

Hussain(19) has given the following definition of a coherent structure based on our present day ability to capture the instantaneous vorticity in a flow by means of hot wire anemometry or LDA combined with computer technology:

"A coherent structure is a connected, large-scale turbulent fluid mass with a phase-correlated vorticity over a spatial extent. That is, underlying the three-dimensional random vorticity fluctuations characterizing turbulence, there is an organized component of the vorticity which is phase correlated (i.e. coherent) over the extent of the structure. (Let us call this spatially phase-correlated vorticity the coherent vorticity). The largest spatial extent over which there is coherent vorticity denotes the extent of the coherent structure. Thus, turbulence consists of coherent phase-random (i.e. incoherent) motions; the latter is superimposed on the former and typically extends beyond the boundary of a coherent structure."

It may be worth while noticing that this definition may be put in perspective by the following statement by the same author concerning flow visualization:

"Smoke or dye boundaries in regions sufficiently far from the point of introduction cannot be trusted as reliable boundaries of coherent structures, because marked fluid may not be the coherent vorticity-bearing fluid."

Hussain(19) maintains that this suggested definition is the only one which properly describes the phenomenon. He seems to be one of the few who have also explored this phenomenon experimentally and Fig. 2 is a reproduction of his Fig. 6(19) exhibiting

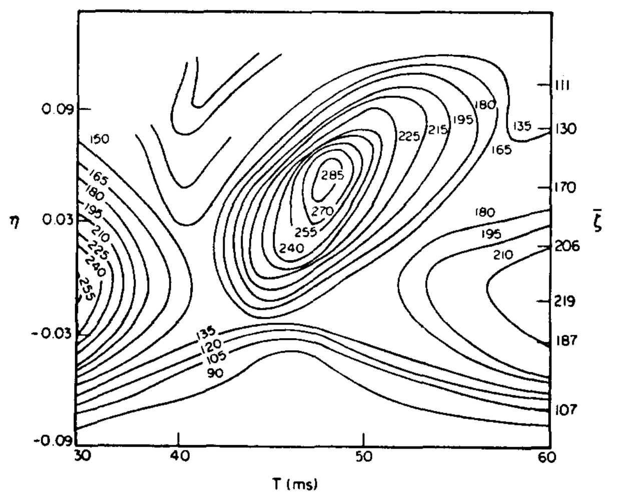

FIG. 2. Coherent vorticity contours of the mixing layer spot at $x/D = 4.5$; $D = 12.7$ cm; $\eta = (y - D)/x$; $U = 20$ m/sec; and T is the time elapsed since spark. Reproduced from Fig. 6 in Hussain.(19)

coherent vorticity contours of a mixing layer spot. This is only one of the many manifestations of coherence in turbulence presented by Hussain. There can thus be no doubt about the existence of coherence.

If coherent structures are to be recognized (in the sense of the definition) in a flow situation, one will have to introduce a 'flow determined time window' over which the averaging process is applied. This is a procedure, comparatively easy to introduce in an experimental setup, where trigger mechanisms can be introduced in the measuring chain which are governed by the flow itself. It is extremely difficult to conceive of an analytic procedure which in this case would replace the Reynolds' assumption (that differential equations for instantaneous quantities will retain their validity after having been subjected to the averaging process) realizing that a selective averaging process (different for different quantities(?)) is to be introduced.

The obvious question to be asked is the one on the applicability of this concept in formulating a 'theory of turbulence'. One may take Hussain's[19] own answer:

> "Accumulated results from many investigations in recent years suggest that coherent structures are characteristic features of many, perhaps all, turbulent shear flows. Since these appear to be associated with significant turbulent transport, any viable turbulence theory must take these into account. No such theory has been proposed, nor is any in sight. Theoreticians should feel encouraged to focus some effort in this direction and utilize the comparatively substantial experimental data generated on coherent structures in a number of turbulent shear flows."

And he goes on to say:

> "Considering all these, it is not at all clear that a collection of deterministic coherent structures can be assumed to represent a shear flow turbulence. While in the initial stages of formation, the structures are dominant and can, therefore, possibly represent most of the total effect to turbulence, these structures do not account for most of turbulence transport in fully developed flows. It does not appear very likely that the coherent structure approach is going to solve the turbulence closure problem or even significantly help in acceptable predictions of technological flows."

The mere fact that these structures and their occurrence depend on the methodology used for their detection makes it almost impossible to conceive of a theory being based on the Navier–Stokes equations and an averaging process which in this way may have to be different for each term in the equation. This is perhaps the largest objection to the approach.

In his exposition of the coherent structures, Hussain[19] considers the well recognized structure of streamwise directed vortices in the boundary layer:

> "The mechanisms for the formation of the counter-rotating vortices and their connection with bursting are quite complex and still not understood. Even though it has been long contended by many that streaks mark the stagnant regions between counter-rotating longitudinal wall vortices, which are responsible for scooping fluid up from the wall and contributing to heat, mass, and momentum transport near the wall and especially in the production of wall shear stress, there is yet no clear understanding of their origin, their role in transporting (spanwise) vorticity from the wall, or even whether they always occur in pairs."

Such structures were discovered and discussed before 1967 by Ginoux,[20] and in a series of reports from 1964–1969 Persen[21] investigated them. One feature is that such vortices are unstable in the sense that they pick up rotational speed with increasing downstream distance and finally burst. Thus, the mechanism responsible for the 'ejection phase' may be seen in the light of this phenomenon and the production of turbulent energy close to the wall may be explained this way even though this cannot be the entire story. A second feature of such structures is their effect on heat transfer from the wall, especially in the region of a stagnation point. This points to the explanation of their creation as a result of the instability caused by curved flow. Persen[22] has discussed this possibility. Thus, several concepts of a physical nature which have been explored experimentally seem not to have found their way into computer codes presently in common use.

6. PROPAGATION OF DISTURBANCES

Identifiable bodies of fluid, 'turbulent balls' (in the Prandl sense?) or coherent structures (of a nature different from Hussain's definition) are produced by the vortex shedding at a jet's nozzle. In spite of the fact that they may differ from Hussain's definition, they certainly contribute to the Reynolds stresses $\rho \overline{u'u'}$ and $\rho \overline{v'v'}$. These 'structures' die out as jets and wakes do. Such a view is also supported by Kline *et al.*[14]:

> "The trajectories of the ejected eddies can be evaluated quantitatively, and this has been done using side view motion pictures (with dye) from the five flows. In each case there is quite a variation among individual trajectories, but by considering a large enough sample a stable average can be obtained at any point. Figure 17 shows the distribution and average trajectories for the zero-pressure-gradient case. Note that the average trajectory and the 'most probable' trajectory are essentially the same, though the variance in trajectories is quite substantial."

This conforms extremely well with the observations of Persen[15] and one may attempt the following conclusion.

A 'disturbance' or a 'coherent structure' dies out as it is being 'transported' by the mean flow by increasing its spread (as the jet's half width) and decreasing its 'intensity' (as the jets centreline velocity) or in other words: the jet's behaviour reflects a general propagation process characteristic of all turbulent flows which are 'free', i.e. uninfluenced by fixed boundaries. The free jet develops independently of Reynolds number, i.e. the viscosity does not significantly enter the process. So do also 'coherent structures',—one had found a typical feature of free turbulence.

This statement can be illustrated by plotting the maximum amplitudes and the half-width (taken from Fig. 3) as functions of distance. Figure 4 shows the result and as predicted the halfwidth is a linear function of downstream distance whereas the centreline amplitude decays according to some power law.

The 'coherent structures' observed at the edge of jet flow originate from the boundary of the nozzle or the shear layer. In boundary layer flow they originate from the shear layer and the wall. In both cases they are characteristic of 'boundary conditions'. The characteristics of the flow is its tendency (or shall we say property of the flow) to let disturbances, once created, 'decay' as described above.

The concepts discussed in the last two paragraphs are relatively new. They have yet failed to have any impact on computational fluid dynamics, mostly because no obvious way of incorporating these new pieces of information into the computational scheme has been found.

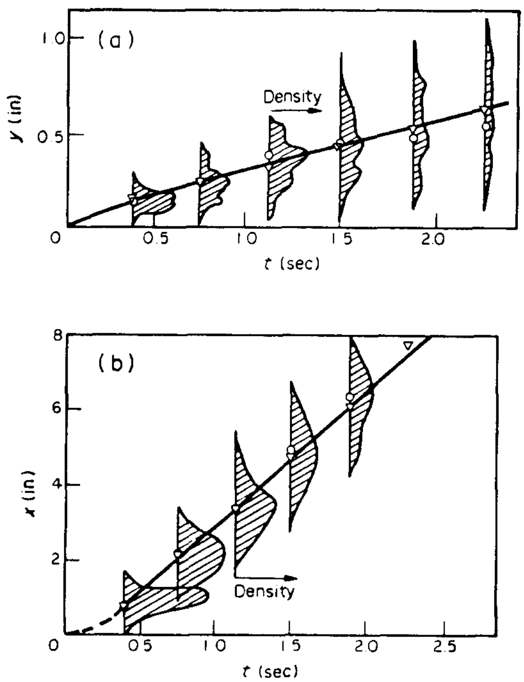

FIG. 3. Trajectories of ejected eddies—flat plate flow. Station 10, $dP/dx = 0$; ∇, averages; $\circ$, maxima. Reproduced from Fig. 17 in Kline *et al.*[14]

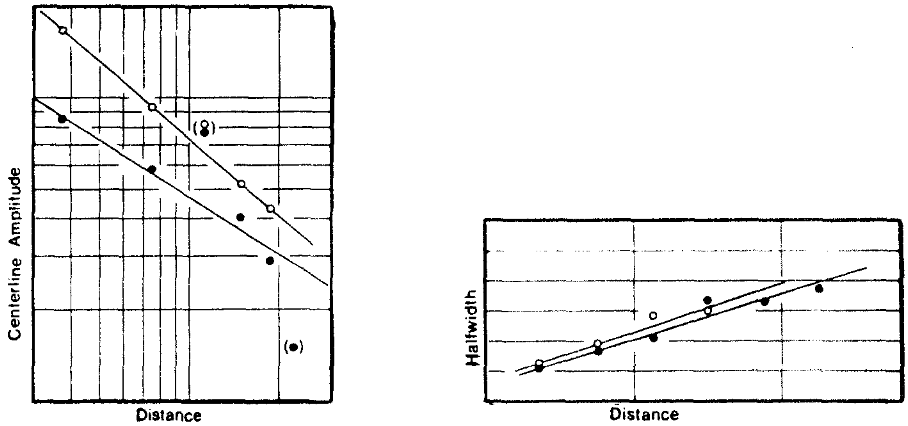

FIG. 4. Centreline amplitude and halfwidth plotted as a function of distance (●, x-direction; ○, y-direction).

7. TURBULENCE MODELS

It has become customary to use the concept 'turbulence models' in connection with the way in which the closure problem is solved. In order to give a varied picture of such models a few typical examples are chosen which more or less cover the main trends.

The first 'model' is the mixing length theory. Considering Prandtl's approach one sees how he approaches the problem directly by seeking a connection between the mean flow field and the Reynolds stresses. Such an approach may be considered to be of zero order, because it does not incorporate any new equations or principles other than those already expressed through Eqs (1) and (2). It may also be considered a straight analogy to the Stokes hypothesis by connecting the state of stress at a point to the state of deformation rate. The flaws in this concept are today well recognized through the fact that the Reynolds stresses are caused by the exchange of momentum across a control surface and not connected with properties of the fluid.

The 'higher order' theories make use of the fact that by manipulation of the basic equations of motion one may establish new equations applying the averaging procedure of Reynolds to them. Thus 'transport' equations both for the turbulent energy and the Reynolds stresses may be obtained. However, conceiving of the Reynolds stresses as one-point double correlations, one will find that the above procedure introduces higher order correlations thus the 'closure' problem is still with us. It is a question which may be debated whether or not the introduction of closure assumption in higher order equations really constitutes an improvement over an introduction at a lower level. The ideas which one meets in this connection will here be exemplified by considering a few typical cases.

The *KOLMOGOROV Model* of 1942 was brought to our attention by Spalding[23] as late as in 1968. It is interesting because it introduces the kinetic energy

$$\tfrac{1}{2}q^2$$

contained in the fluctuating motion (called turbulent energy) as a 'parameter'. In ordinary notation his equation of motion may be expressed as:

$$\frac{D\bar{u}_i}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x_i} - \frac{\partial}{\partial x_j}(\overline{u_i'u_j'}) \tag{7}$$

It is noticed that the viscous stresses are neglected in accordance with Prandtl's original idea that the Reynold's stresses are the dominant ones. These stresses are first modelled on the Stoke's hypothesis

$$\overline{u_i'u_j'} = -\frac{A}{\omega}q^2\left[\frac{\partial \bar{u}_i}{\partial x_j} + \frac{\partial \bar{u}_j}{\partial x_i}\right] + \frac{2}{3}\left[\frac{q^2}{2}\right]\delta_{ij} \tag{8}$$

where it is observed that the $A\overline{q^2}/\omega$ represents the 'turbulent transport coefficient' or the 'eddy viscosity'. In this expression $\overline{q^2}$ and ω are unknown and relations for them are given as follows:

$$\frac{D}{Dt}\left(\frac{1}{2}\overline{q^2}\right)=\frac{A}{\omega}\overline{q^2}\left[\frac{\partial \bar{u}_i}{\partial x_j}+\frac{\partial \bar{u}_j}{\partial x_i}\right]-\omega\frac{\overline{q^2}}{2}+A''\frac{\partial}{\partial x_j}\left[\frac{\overline{q^2}}{2\omega}\frac{\partial}{\partial x_j}\left(\frac{1}{2}\overline{q^2}\right)\right], \tag{9}$$

$$\frac{D\omega}{Dt}=-\frac{7}{10}\omega^2+A'\frac{\partial}{\partial x_j}\left[\frac{\overline{q^2}}{2\omega}\frac{\partial \omega}{\partial x_j}\right]. \tag{10}$$

These equations represent the mathematical formulation of the model. They contain the three constants A, A' and A'' which supposedly are to be determined from comparison with experiments. The two Eqs (9) and (10) are formulated on the basis of analogies and *not* as deductions from the basic equations. It is also easily recognized that the basic idea of Prandtl of expressing a flux as a turbulent transport coefficient times a gradient has found extensive use.

The *PRANDTL MODEL* of 1945 was elaborated upon by Emmons[24] but is mainly aimed at two-dimensional shear flows. The basic equations of this model may be expressed as follows:

continuity:

$$\frac{\partial \bar{u}_i}{\partial x_i}=0 \tag{11}$$

momentum:

$$\frac{D\bar{u}_i}{Dt}=-\frac{1}{\rho}\frac{\partial \bar{p}}{\partial x_i}+\nu\frac{\partial^2 \bar{u}_i}{\partial x_j^2}-\frac{\partial \overline{u_i'u_j'}}{\partial x_j} \tag{12}$$

turbulent kinetic energy:

$$\frac{D(\overline{q^2}/2)}{Dt}=-\overline{u_i'u_j'}\frac{\partial \bar{u}_i}{\partial x_j}-\frac{\partial}{\partial x_j}\left[\frac{\overline{u_j'q^2}}{2}-\frac{\overline{u_j'p'}}{\rho}\right]+\nu\frac{\partial^2(\overline{q^2}/2)}{\partial x_j^2}-\nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)^2} \tag{13}$$

The equation of continuity (11) as well as the three equations of motion (12) have not been subjected to any simplification. The equation for the kinetic energy in the turbulent motion is seen to contain both the one-point triple correlation as well as the double correlation between the pressure and the velocity fluctuations. The last term in this equation deserves special attention. It represents the dissipation and as such does not naturally enter a mechanical energy equation. It occurs because it is added and subtracted to enable rewriting the last but one term in the form given. This point will be discussed in detail later.

The 'closure' problem inherent in the formulation of (11)–(13) is now solved by a certain type of 'modelling'.

(1) The first term on the righthand side of (13) is recognized as a momentum transport term (momentum flux per unit mass). It is 'modelled' as a turbulent transport coefficient times the gradient to the mean flow field. The following assumptions are made:

the 'turbulent transport coefficient':

$$\alpha l\sqrt{\tfrac{1}{2}\overline{q^2}}$$

the mean velocity gradient:

$$\partial \bar{u}/\partial y$$

and consequently:

$$-\overline{u_i'u_j'}=\alpha l\sqrt{\tfrac{1}{2}\overline{q^2}}\,\frac{\partial \bar{u}}{\partial y}. \tag{14}$$

(It is to be realized that now $\bar{u}$ and $\bar{y}$ have been introduced to denote the mean velocity component and the coordinate normal to it respectively.)

(2) The second term on the righthand side of (13) is recognized as a convective diffusion term and again it is 'modelled' the usual way:

the 'turbulent transport coefficient':

$$\beta l \sqrt{\tfrac{1}{2}\overline{q^2}}$$

the 'concentration' gradient:

$$\partial(\tfrac{1}{2}\overline{q^2})/\partial y$$

and consequently:

$$-\frac{\partial}{\partial x_j}\left[\frac{\overline{u_j' q^2}}{2}-\frac{\overline{u_j' p'}}{\rho}\right]=\frac{\partial}{\partial y}\left[\beta l \sqrt{\tfrac{1}{2}\overline{q^2}}\frac{\partial}{\partial y}(\tfrac{1}{2}\overline{q^2})\right]. \tag{15}$$

(3) The third term on the righthand side of (13) does not need rewriting but the fourth term is 'modelled' on the drag of a 'turbulent ball'. A 'ball' of average dimension l may have a velocity proportional to

$$\sqrt{\tfrac{1}{2}\overline{q^2}}$$

and will experience a drag force proportional to

$$l^2 q^2/2.$$

The dissipated energy will be proportional to the work done by this drag force as the 'ball' moves, thus

$$-\nu\overline{\left(\frac{\partial u_i'}{\partial x_j}\right)}=\gamma(\tfrac{1}{2}\overline{q^2})^{3/2}/l. \tag{16}$$

By utilizing the results (14), (15) and (16) one may rewrite equation (13) which upon introduction of the notation

$$E=\tfrac{1}{2}\overline{q^2}$$

may be formulated

$$\frac{DE}{Dt}=\alpha l\sqrt{E}\left(\frac{\partial\bar{u}}{\partial y}\right)^2+\beta\frac{\partial}{\partial y}\left(l\sqrt{E}\frac{\partial E}{\partial y}\right)+\nu\frac{\partial^2 E}{\partial y^2}-\gamma E^{3/2}/l \tag{17}$$

where α, β and γ are constants to be determined. They ought to be universally valid but of course they are not. The 'most energetic eddy size' l must also be determined and has unfortunately no universal character. Emmons[24] used

$$l=3y[1-y/2A_1] \tag{18}$$

for this length scale, an expression deduced from experimental evidence with generous allowance for deviations.

The two models reviewed in detail so far have been chosen to exhibit the concepts and ideas usually applied in the attempts to solve the closure problem. It is remarkable how the simple ideas introduced originally by Prandtl and Komolgorov are still used in attempts to bridge the gaps in our knowledge. It is, however, from a theoretical point of view a rather dangerous path to embark upon because the 'models' may after some time inherit a significance as if they represent a physical reality. And this is where a closer scrutiny of the concepts behind 'modelling' may be justified.

The basic equations of turbulence theories are for the most part deduced by manipulating the equations of motion. Thus, the mechanical energy equation is obtained by multiplying the equations of motion by the velocity components and adding them together. This equation does not contain any new information. It only represents a mathematical consequence of the equations of motion and will be identically satisfied once a solution to these equations is known. In this respect is has its analog in the Bernoulli equation (flow along a streamline) or the von Kármán integral momentum equation. Just as these are given physical significance through proper modification, so the mechanical energy equation acquires its fundamental importance through 'modelling'. However, modelling must be based on a sound physical concept just as in the two analogous cases.

Now, going back to the Prandtl/Emmons model just discussed, one may recognize that the terms to be modelled are first identified as terms in the equation responsible for certain physical processes ('momentum transport term', 'convective diffusion term', 'energy dissipation term', etc.) A successful modelling can only be expected to work if these concepts are sound. So, the recognized failure of the k–ε model (k = turbulent energy, ε = dissipation) to work for recirculating flows must therefore be sought in the concepts used when modelling the terms.

Using the usual mechanistic concepts one may easily show that the mechanical energy equation represent the fundamental relation whereby the rate of change of the kinetic and potential energy of a body (element) of fluid is equal to the work done by the forces acting on the body as it moves. Since the sum of kinetic and potential energy of an element of fluid may both increase and decrease the conclusion is inevitable: the work done by the forces may be both positive and negative. Since this may happen also when only viscous forces are active one must arrive at the conclusion that the viscous forces may produce 'recoverable' work. In other words, in cases where streamlines are closed contours (recirculation) the work done as one goes along the contour and returns to the starting point must be both positive and negative and finally end up as zero at the completion of one round.

A basic remark must be made. The appearance of a dissipation term in the mechanical energy Eq. (13) is artificial. It is introduced artificially by adding and subtracting it whereby the added portion is absorbed in the term $\nu\partial^2(\overline{q^2}/2)\partial x_j^2$ and the subtracted term appears explicitly. Dissipation is always conversion of mechanical energy into heat and as such cannot appear in the mechanical energy equation; it belongs in the formulation of the 1st law of thermodynamics.

The result of these considerations is that a mechanical energy equation with the energy dissipation term in it can be expected to work successfully in developing flows (boundary layers, jets, wakes etc.) but that the physical concepts behind the modelling prevents meaningful results from being obtained when recirculation is a basic feature of the flow. Even though the k–ε-model is extremely well deduced by Rodi and Rastogi[25] who apply the more sophisticated model in Eq. (8) for the Reynolds stresses, it cannot escape the previous conclusions: (1) the concept of one turbulent transport quantity valid for all nine components of the tensor is very simplistic and is probably the greatest obstacle to general validity of the approach, (2) the fact that energy is assumed always dissipating along a streamline is an obstacle to its validity in the case of recirculating flows. (In this context it may be an interesting exercise to contemplate the head loss when moving from a fixed point in a closed loop back to the origin in a pipeline network.)

In a detailed exhibition of the turbulence modelling necessary to obtain an approach which will handle recirculating flow Boysan and Johansen[26] show impressive numerical results for such flows. However, even if transport equations for the Reynolds stresses are introduced, the necessity to rid the equations of higher order correlations makes drastic assumptions necessary.

Perhaps the most ambitious attempt at establishing a complete theory of turbulence was made by Rotta.[27] If the PRANDTL mixing length theory is of the zero order and his 1945 model of the first order (1 transport equation for $\overline{q^2}$) then *ROTTA's model* is of the 7th order (7 transport equations: for $\overline{q^2}$, for the length scale l and for the Reynolds stresses $\overline{u_i'u_j'}$). The higher the order of the theory one tries to establish the more complex becomes the task of finding proper closure models since the physics tends to involve a massive mathematical complexity.

Having now, perhaps in a somewhat sketchy manner, exhibited how ideas and concepts are generally introduced into the basic equations which serve as a basis for numerical schemes, the time has come to ask some crucial questions. Have we a closure problem because we have avoided the real issue? We know that the fundamental conservation laws must be supplemented by proper phenomenological relations which are experimental in nature. Since manipulation of equations only leads to new equations which require an increased amount of needed phenomenological information one may wonder if higher order theories really can be considered as 'improvements'.

A final note ought to be struck. At the *Third Symposium on Numerical and Physical Aspects of Aerodynamic Flows*, January 1985, Long Beach, California, 16 papers discussing turbulent modelling were given. Out of these, nine used the Cebeci–Smith model (extensively covered by Carr and Cebeci[28]) and five used the k–ε model. At the *International Symposium on Refined Flow Modelling and Turbulence Measurements*, September 1985, IOWA City, IOWA, a test case was presented by Bouffinier and Grandotta[29] where 17 contributors had computed the flow using the k–ε model. The results were confronted with experimental evidence, and velocity profiles as well as distribution of turbulent kinetic energy were used as criteria for accuracy. The following is quoted from the summary.

> "Concerning the mean velocity field quite all the results compared well together and with the experiment. Several calculations gave similar results for the turbulent quantities, but all differed clearly from the measurements."

This indicates how careful one ought to be when selecting a criterion for the success of a computer code, but it also warns about putting too much faith in beautifully prepared computer produced flow fields. They do not always reflect reality.

In an extensive review paper (63 references) McCroskey *et al.*[30] concentrates on the prediction of nonsteady turbulent flows by means of computer codes. It is interesting to notice the following paragraphs from the summary:

> "Except for the limitations of turbulence modelling, the showcase problems of 1974 can be solved routinely today."

> "The validity (of numerical simulations) is essentially determined by the turbulence modelling, ..."

> "The area of turbulence modelling is probably the one with the least optimism, . . ."

So, the time has come to look for new ideas which in the author's opinion can only come from a scrutiny of existing (and perhaps new improved) experimental evidence. Support for such an opinion is again presented by McCroskey *et al.*[30]

> "This goal will be achieved, however, only with the aid of high quality physical experiments. That is, detailed experiments will have to play crucial roles in improving the turbulence and vortex modelling and in guiding and validating the numercial simulations, whatever their levels of complexity."

This ought to present our experimentors with a great challenge.

8. 'EXACT' SOLUTIONS

At this stage one may stop and ask a different question. 'With all the information hidden in the numerous experimental investigations available, have we no information available which yields a case which can be solved from first principles?' The answer is a conditional 'yes'. The condition is that one must simplify the flow to that of a boundary layer on a flat plate. In that case, and accepting a certain level of accuracy, one may use the information contained in (4) as follows. In general $\bar{u} = F(x,y)$ will be the solution. This may, by a change of variables and with no loss of generality, be reformulated as:

$$u^+ = F_1(x, y^+) \tag{19}$$

which Eq. (4) and experimental evidence (within a certain degree of accuracy) show can be simplified to

$$y^+ = \phi(u^+) \tag{20}$$

known as the law of the wall. The fact that no explicit x-dependence appears in the law of the wall is actually a strong enough piece of phenomenological information to permit a solution from first principle.[31] However, a scrutiny of the results reveals the following shortcoming of the resulting solution:

(1) The velocity profile at the outer edge of the boundary layer exhibits a kink.
(2) The influence of outer manipulation like pressure gradient or 'history' is not incorporated in the approach.

The remedy to this is rather easily found and is exhibited in the procedure usually characterized as the 'method of inner variables'.[32] This procedure reveals however the importance of being able to incorporate the 'history' in the initial or boundary conditions under which the solution is to be sought. This point seems very often to be overlooked in ordinary expositions of boundary layer theory.

One feature stands out in this connection, a feature which ought to be properly incorporated in numerical analysis of turbulent flows. A scrutiny of the central cases contained in the proceedings of the *Stanford Conference* on turbulent boundary layers[33] reveals[32] that a wall region exists which is uninfluenced by outside manipulation. Such manipulation influences only the law of the wake and determines how close to the wall this influence is felt. The Spalding[34] formulation of the law of the wall may be used

$$y^+ = u^+ + A\left[e^{\kappa u^+} - 1 - \kappa u^+ - (\kappa u^+)^2/2 - (\kappa u^+)^3/6 - (\kappa u^+)^4/24\right] \tag{21}$$

where

$$A = 0.015, \quad \kappa = 0.53227. \tag{22}$$

A complete record of this can be found in Ref. (35).

This is really a showcase where experimental evidence is drawn upon to furnish the additional phenomenological information to establish a complete theory rendering a solution from first principles. This information occurs in an unexpected form (law of the wall) and is proved to be one way of settling the closure problem. Thus one will not necessarily have to look for stress/strain rate type of phenomenological information. The realization of this fact may perhaps trigger off invention of unexpected experimentally based relations which can lead out of the dilemma.

9. FLOW FUNCTIONS

In connection with the closing statement of the last paragraph, attention is drawn to a different point of view when considering the closure problem. The eqs. (2) are reformulated as:

$$\left.\begin{aligned}
&\frac{\partial}{\partial x}[\sigma_x - p - \rho\overline{u'u'} - \rho\bar{u}^2] + \frac{\partial}{\partial y}[\tau_{xy} - \rho\overline{u'v'} - \rho\bar{u}\,\bar{v}] + \frac{\partial}{\partial z}[\tau_{xz} - \rho\overline{u'w'} - \rho\bar{u}\bar{w}] = 0\\
&\frac{\partial}{\partial x}[\bar{\tau}_{xy} - \rho\overline{u'v'} - \rho\bar{u}\,\bar{v}] + \frac{\partial}{\partial y}[\bar{\sigma}_y - p - \rho\overline{v'v'} - \rho\bar{v}^2] + \frac{\partial}{\partial z}[\bar{\tau}_{yz} - \rho\overline{v'w'} - \rho\bar{v}\bar{w}] = 0\\
&\frac{\partial}{\partial x}[\bar{\tau}_{xz} - \rho\overline{u'w'} - \rho\bar{u}\bar{w}] + \frac{\partial}{\partial y}[\bar{\tau}_{yz} - \rho\overline{v'w'} - \rho\bar{v}\bar{w}] + \frac{\partial}{\partial z}[\bar{\sigma}_z - p - \rho\overline{w'w'} - \rho\bar{w}^2] = 0
\end{aligned}\right\} \tag{23}$$

These are the equations of dynamic equilibrium in the d'Alembert's sense, where the terms of the type $\rho\bar{u}^2$, etc. represent the inertia forces which here appear as stresses. These equations of dynamic equilibrium will be identically satisfied if one introduces the flow function ψ and the 3 potential functions ϕ_1, ϕ_2 and ϕ_3 such that:

$$\left.\begin{aligned}
\bar{\sigma}_x-\bar{p}-\rho\overline{u'u'}-\rho\bar{u}^2&=\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}+\alpha\left(\frac{\partial\phi_1}{\partial x}-\frac{\partial\phi_2}{\partial y}-\frac{\partial\phi_3}{\partial z}\right)\\
\bar{\sigma}_y-\bar{p}-\rho\overline{v'v'}-\rho\bar{v}^2&=\frac{\partial^2\psi}{\partial z^2}+\frac{\partial^2\psi}{\partial x^2}+\alpha\left(\frac{\partial\phi_2}{\partial y}-\frac{\partial\phi_3}{\partial z}-\frac{\partial\phi_1}{\partial x}\right)\\
\bar{\sigma}_z-\bar{p}-\rho\overline{w'w'}-\rho\bar{w}^2&=\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\alpha\left(\frac{\partial\phi_3}{\partial z}-\frac{\partial\phi_1}{\partial x}-\frac{\partial\phi_2}{\partial y}\right)
\end{aligned}\right\}\tag{24}$$

and

$$\left.\begin{aligned}
\bar{\tau}_{xy}-\rho\overline{u'v'}-\rho\bar{u}\,\bar{v}&=-\frac{\partial^2\psi}{\partial x\partial y}+\alpha\left(\frac{\partial\phi_1}{\partial y}+\frac{\partial\phi_2}{\partial x}\right)\\
\bar{\tau}_{yz}-\rho\overline{v'w'}-\rho\bar{v}\bar{w}&=-\frac{\partial^2\psi}{\partial y\partial z}+\alpha\left(\frac{\partial\phi_2}{\partial z}+\frac{\partial\phi_3}{\partial y}\right)\\
\tau_{zx}-\rho\overline{u'w'}-\rho\bar{u}\bar{w}&=-\frac{\partial^2\psi}{\partial x\partial z}+\alpha\left(\frac{\partial\phi_3}{\partial x}+\frac{\partial\phi_1}{\partial z}\right).
\end{aligned}\right\}\tag{25}$$

This is in complete accordance with what Neuber[36] calls his 'three-functions-approach', and it is easily shown that if (24) and (25) are satisfied, the three equations of dynamic equilibrium are identically satisfied provided

$$\nabla^2\phi_1=0,\qquad \nabla^2\phi_2=0,\qquad \nabla^2\phi_3=0. \tag{26}$$

One may now use this approach to investigate the two-dimensional case in which $\bar{w}\equiv 0$ and $\partial\bar{u}/\partial\bar{z}=\partial\bar{v}/\partial\bar{z}\equiv 0$. The situation will then be such that the only mean stress components which are different from zero will be $\bar{\sigma}_x$, $\bar{\sigma}_y$ and $\bar{\tau}_{xy}$. Thus one will have the following expression for the Reynolds apparent stresses:

$$\left.\begin{aligned}
-\rho\overline{u'u'}&=\rho\bar{u}^2-2\mu\frac{\partial\bar{u}}{\partial x}+\bar{p}+\frac{\partial^2\psi}{\partial y^2}+\frac{\partial^2\psi}{\partial z^2}+\alpha\left(\frac{\partial\phi_1}{\partial x}-\frac{\partial\phi_2}{\partial y}-\frac{\partial\phi_3}{\partial z}\right)\\
-\rho\overline{v'v'}&=\rho\bar{v}^2-2\mu\frac{\partial\bar{v}}{\partial y}+\bar{p}+\frac{\partial^2\psi}{\partial z^2}+\frac{\partial\psi^2}{\partial x^2}+\alpha\left(\frac{\partial\phi_2}{\partial y}-\frac{\partial\phi_3}{\partial z}-\frac{\partial\phi_1}{\partial x}\right)\\
-\rho\overline{w'w'}&=\qquad\qquad\quad +\bar{p}+\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^2\psi}{\partial y^2}+\alpha\left(\frac{\partial\phi_3}{\partial z}-\frac{\partial\phi_1}{\partial x}-\frac{\partial\phi_2}{\partial y}\right)
\end{aligned}\right\}\tag{27}$$

and

$$\left.\begin{aligned}
-\rho\overline{u'v'}&=\rho\bar{u}\,\bar{v}-\left(\frac{\partial\bar{u}}{\partial y}+\frac{\partial\bar{v}}{\partial x}\right)-\frac{\partial^2\psi}{\partial x\partial y}+\alpha\left(\frac{\partial\phi_1}{\partial y}+\frac{\partial\phi_2}{\partial x}\right)\\
-\rho\overline{v'w'}&=\qquad\qquad\qquad\quad -\frac{\partial^2\psi}{\partial y\partial z}+\alpha\left(\frac{\partial\phi_2}{\partial z}+\frac{\partial\phi_3}{\partial y}\right)\\
-\rho\overline{w'u'}&=\qquad\qquad\qquad\quad -\frac{\partial^2\phi}{\partial x\partial z}+\alpha\left(\frac{\partial\phi_3}{\partial x}+\frac{\partial\phi_1}{\partial z}\right)
\end{aligned}\right\}\tag{28}$$

These equations reveal how the Reynolds' stresses $\rho\overline{w'w'}$, $\rho\overline{v'w'}$ and $\rho\overline{u'w'}$ may vary with the space coordinates (x,y,z) even in a situation which offhand seems to be purely two-dimensional. If one adds the condition, that all derivatives with respect to z vanish, one obtains the following expressions for the Reynolds' stresses usually considered in a two-dimensional case

$$\left.\begin{aligned}
-\rho\overline{u'u'} &= \rho\bar{u}^2 - 2\mu\frac{\partial\bar{u}}{\partial x} + p + \frac{\partial^2\psi}{\partial y^2} + \alpha\left(\frac{\partial\phi_1}{\partial x} - \frac{\partial\phi_2}{\partial y}\right)\\
-\rho\overline{v'v'} &= \rho\bar{v}^2 - 2\mu\frac{\partial\bar{v}}{\partial y} + p + \frac{\partial^2\psi}{\partial x^2} + \alpha\left(\frac{\partial\phi_2}{\partial y} - \frac{\partial\phi_1}{\partial x}\right)\\
-\rho\overline{u'v'} &= \rho\bar{u}\,\bar{v} - \mu\left(\frac{\partial\bar{u}}{\partial y} + \frac{\partial\bar{v}}{\partial x}\right) - \frac{\partial^2\psi}{\partial x\partial y} + \alpha\left(\frac{\partial\phi_1}{\partial y} + \frac{\partial\phi_2}{\partial x}\right)
\end{aligned}\right\} \qquad (29)$$

One is here in the position to conceive of the flow function ψ as a modified Airy's stress function for the Reynolds apparent stresses in a two-dimensional turbulent flow.

The relations exhibited here are different from what usually is being presented in connection with turbulence modelling. Could it be that one should seek experimental evidence on the connection between the matrices

$$\bar{u}_i\bar{u}_j \text{ and } \overline{u_i'u_j'}$$

as indicated by Eq. (23).

10. CHAOS

The later years have seen the 'theory of chaos' entering the turbulence scene. A remarkable paper has been presented by Cheng[37] in which not only the implications of the strange attractors* are brought out, but where also a more or less philosophical confrontation with the statistical approach with its ergodic principle is given. It is tempting to draw attention to the following quotation from this paper:

"When initial boundary data are specified, the strange attractor theory claims that the repeated (and/or repeating) bifurcations of the solution of the NS will give a chaotic state deterministically, which is fluid turbulence. If the initial data are perturbed, the details of the chaotic state will change significantly. The statistics will depend upon the initial data. Ensemble average will be ensemble specific. Fluid turbulence will be 'statistical' only in so far as the initial boundary data of the flow are uncertain. Turbulence modelling will inevitably be of limited utility, i.e. valid only within the ensemble."

There is mentioned in this quotation the possibility that turbulence is deterministic, that the fluid flow usually called turbulent is a deterministic solution to the Navier–Stokes equations as indicated initially as an almost generally accepted fact. Turbulence should then be a very complex solution with such features that it could be confused with a stochastic system.

Cheng[37] goes on to criticise the usual approach in statistical theory whereby major use is made of the 'gradient type diffusion' as already mentioned in the discussion of the modelling of turbulence. His criticism on this point is however more aimed at the misconceptions introduced through modelling and not so much at the basic concept of the statistical approach. In the end the strange attractor theory is also scrutinized as the following quotation shows.

"Despite the various aspects mentioned above favoring the SA theory, an important weakness exists in its development even if it need not directly support the classical statistical theory of fluid turbulence. The structural aspects of SA are inferred largely from numerical studies of solution bifuration based on simple models, often as initial value problems of some system of ordinary differential equations. These results need not apply to the chaotic fields generated by the much more complex partial differential equations system like Navier–Stokes for general fluid dynamics."

It would lead too far to go into detail in this matter. In an exposition of concepts and ideas in numerical fluid dynamics it is however interesting to notice that the numerical

*Definition taken from Cheng[37]:
"By 1970, Ruelle and Takens[38] showed mathematically that chaos result from repealed solution bifurcation of the initial boundary value problem of a non-linear equation with some sufficienty large parameters like Reynolds number. They call such chaotic solutions the Strange Attractors (SA) to distinguish from the ordinary attractors of fixed points and limit cycles in solution phase space."

procedures may, through their inherent iterative process, introduce chaos, i.e. one has found 'numerical turbulence', not necessarily with physical relevance.

For one who comes from the laboratory where the new developments in computer capability (speed) as well as in the electronic equipment available to the experimentor, the basic concepts of statistical theory are very much a reality. The data handling makes it possible for the experimentor rather rapidly to assure himself that his measurements are not 'ensemble specific', at least not to a degree that makes his measurements unfit for practical purposes. And in the end this is all that we need to know.

At this point we have arrived at the stage where discussions are still going on, where the probing of nature is at its most interesting stage and where perhaps new ideas for better turbulent modelling may be expected to grow.

11. FINAL APOLOGIES

This presentation suffers from a series of deficiencies and omissions for which the author has to apologize. The chaos theory is at present mostly centred on the road to chaos and not so much on what happens next. The coherent structures have become a topic for rather heated discussions but the associated ideas have not yet had any widespread impact on numerical computer codes although they may have brought new information on what turbulent fluid flow is all about.

A final note of uneasiness must be struck. The present exposition has drawn attention very much to the deficiencies of turbulence modelling emphasized by several authors, and joined by the present author. However, the performance of many computer codes seems to be rather good in accurately predicting flow patterns and forces. A question then inevitably comes to mind: Is turbulence modelling really such a crucial matter or can we cope just as well with a crude model without attempting to gain a better physical understanding? The present article is meant to inspire further efforts against such a defeatist view.

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